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TRIGONOMETRY, SURVEYING AND NAVIGATION

BY

G. A. WENTWORTH
AUTHOR OF A SERIES OF TEXT-BOOKS IN MATHEMATICS

SECOND REVISED EDITION

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PREFACE

In preparing this work the aim has been to furnish just so much of Trigonometry as is actually taught in our best schools and colleges. Consequently, all investigations that are important only for the special student have been omitted, except the development of functions in series. The principles have been unfolded with the utmost brevity consistent with simplicity and clearness, and interesting problems have been selected with a view to awaken a real love for the study. Much time and labor have been spent in devising the simplest proofs for the propositions, and in exhibiting the best methods of arranging the logarithmic work.

The object of the work on Surveying and on Navigation is to present these subjects clearly and in so small a compass that students in general may find the time to acquire a competent knowledge of these important studies.

The author is under obligation to G. A. Hill, A.M., of Cambridge, Mass.; to Professor J. L. Patterson, of Chestnut Hill, Pa.; to Dr. F. N. Cole, of New York, N.Y.; to Professor S. F. Norris, of Baltimore, Md.; and to Professor B. F. Yanney, of Alliance, Ohio. Professor Yanney has done most of the work in the revision of the Surveying. Miss M. Gertrude Cross, of Boston, Mass., has furnished the drawings for this edition.

G. A. WENTWORTH.

Exeter, N.H., 1903.

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$a = 40,\ b = 27.$

$\frac{4}{2}\frac{0}{7} = 1.4815$

$\tan A = 1.4815$

$\therefore A = 55° 59'$

$B = 34° 1'$

$a^2 = 1600$

$b^2 = \ 729$

$c^2 = 2329$

$\therefore c = \sqrt{2329}$

$= 48.26$

SECTION XII

GENERAL METHOD OF SOLVING THE RIGHT TRIANGLE

From these five cases it appears that the general method of finding an unknown part in a right triangle is as follows:

Choose from the equation A + B = 90°, *and the equations that define the functions of the angles, an equation in which the required part only is unknown; solve this equation, if necessary, to find the value of the unknown part; then compute the value.*

NOTE. In Case IV, if the given sides (here a and c) are nearly alike in value, then A is near 90°, and its value cannot be accurately found from the tables, because the sines of large angles differ little in value (as is evident from Fig. 4). In this case it is better to find B first, by means of the formula given on page 59, namely,

$$\tan \tfrac{1}{2} B = \sqrt{\frac{c-a}{c+a}}.$$

EXAMPLE. Given $a = 49,\ c = 50$; find A, B, b.

$c - a = 1,\ c + a = 99.$

$\frac{c-a}{c+a} = 0.01010$

$\sqrt{\frac{c-a}{c+a}} = 0.1005$

$\tan \frac{1}{2} B = 0.1005$

$\therefore \frac{1}{2} B = 5° 44'$

$B = 11° 28'$

$A = 78° 32'$

$c - a = 1$

$c + a = 99$

$c^2 - a^2 = 99$

$b^2 = 99$

$b = \sqrt{99}$

$= 9.95$

EXERCISE VIII

1. In Case II give another way of finding c, after b has been found.

2. In Case III give another way of finding c, after a has been found.

3. In Case IV give another way of finding b, after the angles have been found.

4. In Case V give another way of finding c, after the angles have been found.

5. Given B and c; find A, a, b.

6. Given B and b; find A, a, c.

7. Given B and a; find A, b, c.

8. Given b and c; find A, B, a.

Solve the following right triangles:

	Given	Required
9	$a = 3$, $b = 4$.	$A = 36° 52'$, $B = 53° 8'$, $c = 5$.
10	$a = 7$, $c = 13$.	$A = 32° 35'$, $B = 57° 25'$, $b = 10.954$.
11	$a = 5.3$, $A = 12° 17'$.	$B = 77° 43'$, $b = 24.342$, $c = 24.918$.
12	$a = 10.4$, $B = 43° 18'$.	$A = 46° 42'$, $b = 9.800$, $c = 14.290$.
13	$c = 26$, $A = 37° 42'$.	$B = 52° 18'$, $a = 15.900$, $b = 20.572$.
14	$c = 140$, $B = 24° 12'$.	$A = 65° 48'$, $a = 127.694$, $b = 57.386$.
15	$b = 19$, $c = 23$.	$A = 34° 18'$, $B = 55° 42'$, $a = 12.961$.
16	$b = 98$, $c = 135.2$.	$A = 43° 33'$, $B = 46° 27'$, $a = 93.139$.
17	$b = 42.4$, $A = 32° 14'$.	$B = 57° 46'$, $a = 26.733$, $c = 50.124$.
18	$b = 200$, $B = 46° 11'$.	$A = 43° 49'$, $a = 191.900$, $c = 277.160$.
19	$a = 95$, $b = 37$.	$A = 68° 43'$, $B = 21° 17'$, $c = 101.951$.
20	$a = 6$, $c = 103$.	$A = 3° 21'$, $B = 86° 39'$, $b = 102.825$.
21	$a = 3.12$, $B = 5° 8'$.	$A = 84° 52'$, $b = 0.280$, $c = 3.133$.
22	$a = 17$, $c = 18$.	$A = 70° 48'$, $B = 19° 12'$, $b = 5.916$.
23	$c = 57$, $A = 38° 29'$.	$B = 51° 31'$, $a = 35.471$, $b = 44.620$.
24	$a + c = 18$, $b = 12$.	$A = 22° 37'$, $B = 67° 23'$, $a = 5$, $c = 13$.
25	$a + b = 9$, $c = 8$.	$A = 82° 18'$, $B = 7° 42'$, $\begin{cases} a = 7.928, \\ b = 1.072. \end{cases}$

SECTION XIII

SOLUTION BY LOGARITHMS

Case I

Given $A = 34° 28'$, $c = 18.75$; find B, a, b.

1. $B = 90° - A = 55° 32'$.
2. $\frac{a}{c} = \sin A$; $\therefore a = c \sin A$.
3. $\frac{b}{c} = \cos A$; $\therefore b = c \cos A$.

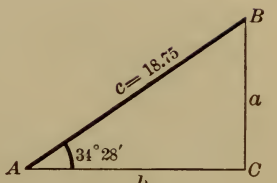


Fig. 15

$\log a = \log c + \log \sin A$	$\log b = \log c + \log \cos A$
$\log c = 1.27300$	$\log c = 1.27300$
$\log \sin A = \underline{9.75276 - 10}$	$\log \cos A = \underline{9.91617 - 10}$
$\log a = 1.02576$	$\log b = 1.18917$
$a = 10.611$	$b = 15.459$

Case II

Given $A = 62° 10'$, $a = 78$; find B, b, c.

1. $B = 90° - A = 27° 50'$.
2. $\frac{b}{a} = \cot A$; $\therefore b = a \cot A$.
3. $\frac{a}{c} = \sin A$.

$\therefore a = c \sin A$, and $c = \frac{a}{\sin A}$.

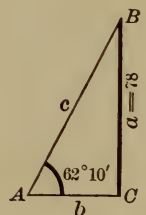


Fig. 16

$\log b = \log a + \log \cot A$	$\log c = \log a + \operatorname{colog} \sin A$
$\log a = 1.89209$	$\log a = 1.89209$
$\log \cot A = \underline{9.72262 - 10}$	$\operatorname{colog} \sin A = \underline{0.05340}$
$\log b = 1.61471$	$\log c = 1.94549$
$b = 41.182$	$c = 88.204$

Case III

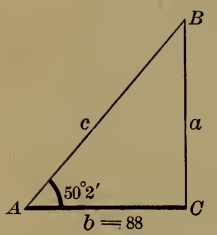


Fig. 17

Given $A = 50°\ 2'$, $b = 88$; find B, a, c

1. $B = 90° - A = 39°\ 58'$.

2. $\frac{a}{b} = \tan A$; $\therefore a = b \tan A$.

3. $\frac{b}{c} = \cos A$.

$\therefore b = c \cos A$, and $c = \frac{b}{\cos A}$.

$$\log a = \log b + \log \tan A$$
$$\log b = 1.94448$$
$$\log \tan A = \underline{10.07670 - 10}$$
$$\log a = 2.02118$$
$$a = 105.00$$

$$\log c = \log b + \operatorname{colog} \cos A$$
$$\log b = 1.94448$$
$$\operatorname{colog} \cos A = \underline{0.19223}$$
$$\log c = 2.13671$$
$$c = 137.00$$

Case IV

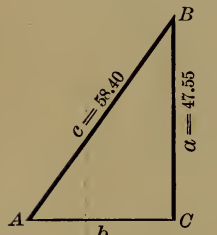


Fig. 18

Given $c = 58.40$, $a = 47.55$; find A, B, b.

1. $\sin A = \frac{a}{c}$.

2. $B = 90° - A$.

3. $\frac{b}{a} = \cot A$; $\therefore b = a \cot A$.

$$\log \sin A = \log a + \operatorname{colog} c$$
$$\log a = 1.67715$$
$$\operatorname{colog} c = \underline{8.23359 - 10}$$
$$\log \sin A = 9.91074 - 10$$
$$A = 54°\ 31'$$
$$B = 35°\ 29'$$

$$\log b = \log a + \log \cot A$$
$$\log a = 1.67715$$
$$\log \cot A = \underline{9.85300 - 10}$$
$$\log b = 1.53015$$
$$b = 33.896$$

Case V

Given $a = 40$, $b = 27$; find A, B, c.

1. $\tan A = \frac{a}{b}$.

2. $B = 90° - A$.

3. $\frac{a}{c} = \sin A$.

$\therefore a = c \sin A$, and $c = \frac{a}{\sin A}$.

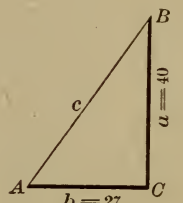


Fig. 19

$$\log \tan A = \log a + \operatorname{colog} b$$

$$\begin{aligned} \log a &= 1.60206 \\ \operatorname{colog} b &= \overline{8.56864 - 10} \\ \log \tan A &= \overline{10.17070 - 10} \\ A &= 55° 59' \\ B &= 34° 1' \end{aligned}$$

$$\log c = \log a + \operatorname{colog} \sin A$$

$$\begin{aligned} \log a &= 1.60206 \\ \operatorname{colog} \sin A &= \overline{0.08151} \\ \log c &= \overline{1.68357} \\ c &= 48.258 \end{aligned}$$

Note. In Cases IV and V the unknown side may also be found from the equations

(for Case IV) $b = \sqrt{c^2 - a^2} = \sqrt{(c + a)(c - a)}$;

(for Case V) $c = \sqrt{a^2 + b^2}$.

These equations express the values of b and c directly in terms of the two given sides; and if the values of the sides are simple numbers (*e.g.*, 5, 12, 13), it is often easier to find b or c in this way. But this value of c is not adapted to logarithms, and this value of b is not so readily found by logarithms as the value of b given under Case IV. See also p. 23.

SECTION XIV

AREA OF THE RIGHT TRIANGLE

The area of a triangle is equal to one-half the product of the base by the altitude; therefore, if a and b denote the legs of a right triangle, and F the area, $F = \frac{1}{2} ab$.

Hence, the area may be found when a and b are known.

For example: Find the area, having given:

CASE I (Sect. XIII, p. 25).

$A = 34° 28'$, $c = 18.75$.

First find (as in Sect. XIII, p. 25) $\log a$ and $\log b$.

$$\log F = \log a + \log b + \operatorname{colog} 2$$
$$\log a = 1.02576$$
$$\log b = 1.18917$$
$$\operatorname{colog} 2 = 9.69897 - 10$$
$$\log F = 1.91390$$
$$F = 82.016$$

CASE IV (Sect. XIII, p. 26).

$a = 47.55$, $c = 58.40$.

First find (as in Sect. XIII, p. 26) $\log a$ and $\log b$.

$$\log F = \log a + \log b + \operatorname{colog} 2$$
$$\log a = 1.67715$$
$$\log b = 1.53015$$
$$\operatorname{colog} 2 = 9.69897 - 10$$
$$\log F = 2.90627$$
$$F = 805.88$$

EXERCISE IX

Solve the following triangles by logarithms, finding the angles to the nearest minute:

	GIVEN		REQUIRED		
1	$a=6$,	$c=12$.	$A=30°$,	$B=60°$,	$b=10.392$.
2	$A=60°$,	$b=4$.	$B=30°$,	$c=8$,	$a=6.9282$.
3	$A=30°$,	$a=3$.	$B=60°$,	$c=6$,	$b=5.1961$.
4	$a=4$,	$b=4$.	$A=B=45°$,	$c=5.6568$.	
5	$a=2$,	$c=2.82843$.	$A=B=45°$,	$b=2$.	
6	$c=627$,	$A=23° 30'$.	$B=66° 30'$,	$a=250.02$,	$b=575.0$.
7	$c=2280$,	$A=28° 5'$.	$B=61° 55'$,	$a=1073.3$,	$b=2011.5$.
8	$c=72.15$,	$A=39° 34'$.	$B=50° 26'$,	$a=45.958$,	$b=55.620$.
9	$c=1$,	$A=36°$.	$B=54°$,	$a=0.58779$,	$b=0.80902$.
10	$c=200$,	$B=21° 47'$.	$A=68° 13'$,	$a=185.72$,	$b=74.22$.
11	$c=93.4$,	$B=76° 25'$.	$A=13° 35'$,	$a=21.936$,	$b=90.788$.
12	$a=637$,	$A=4° 35'$.	$B=85° 25'$,	$b=7946$,	$c=7971.5$.
13	$a=48.532$,	$A=36° 44'$.	$B=53° 16'$,	$b=65.031$,	$c=81.144$.
14	$a=0.0008$,	$A=86°$.	$B=4°$,	$b=0.0000559$,	$c=0.000802$.
15	$b=50.937$,	$B=43° 48'$.	$A=46° 12'$,	$a=53.116$,	$c=73.59$.
16	$b=2$,	$B=3° 38'$.	$A=86° 22'$,	$a=31.496$,	$c=31.559$.

	Given		Required		
17	$a=992$,	$B=76°\ 19'$.	$A=13°\ 41'$,	$b=4074.5$,	$c=4193.5$.
18	$a=73$,	$B=68°\ 52'$.	$A=21°\ 8'$,	$b=188.86$,	$c=202.47$.
19	$a=2.189$,	$B=45°\ 25'$.	$A=44°\ 35'$,	$b=2.2211$,	$c=3.1185$.
20	$b=4$,	$A=37°\ 56'$.	$B=52°\ 4'$,	$a=3.1176$,	$c=5.0714$.
21	$c=8590$,	$a=4476$.	$A=31°\ 24'$,	$B=58°\ 36'$,	$b=7332.8$.
22	$c=86.53$,	$a=71.78$.	$A=56°\ 3'$,	$B=33°\ 57'$,	$b=48.324$.
23	$c=9.35$,	$a=8.49$.	$A=65°\ 14'$,	$B=24°\ 46'$,	$b=3.917$.
24	$c=2194$,	$b=1312.7$.	$A=53°\ 15'$,	$B=36°\ 45'$,	$a=1758$.
25	$c=30.69$,	$b=18.256$.	$A=53°\ 30'$,	$B=36°\ 30'$,	$a=24.67$.
26	$a=38.313$,	$b=19.522$.	$A=63°$,	$B=27°$,	$c=43$.
27	$a=1.2291$,	$b=14.950$.	$A=\ 4°\ 42'$,	$B=85°\ 18'$,	$c=15$.
28	$a=415.38$,	$b=62.080$.	$A=81°\ 30'$,	$B=\ 8°\ 30'$,	$c=420$.
29	$a=13.690$,	$b=16.926$.	$A=38°\ 58'$,	$B=51°\ 2'$,	$c=21.769$.
30	$c=91.92$,	$a=2.19$.	$A=\ 1°\ 22'$,	$B=88°\ 38'$,	$b=91.894$

Compute the unknown parts and also the area, having given:

31. $a=5$, $b=6$.
32. $a=0.615$, $c=70$.
33. $b=\sqrt[3]{2}$, $c=\sqrt{3}$.
34. $a=7$, $A=18°\ 14'$.
35. $b=12$, $A=29°\ 8'$.
36. $c=68$, $A=69°\ 54'$.
37. $c=27$, $B=44°\ 4'$.
38. $a=47$, $B=48°\ 49'$.
39. $b=9$, $B=34°\ 44'$.
40. $c=8.462$, $B=86°\ 4'$.
41. Find the value of F in terms of c and A.
42. Find the value of F in terms of a and A.
43. Find the value of F in terms of b and A.
44. Find the value of F in terms of a and c.
45. Given $F=58$, $a=10$; solve the triangle.
46. Given $F=18$, $b=5$; solve the triangle.
47. Given $F=12$, $A=29°$; solve the triangle.
48. Given $F=100$, $c=22$; solve the triangle.
49. Find the angles of a right triangle if the hypotenuse is equal to three times one of the legs.

50. Find the legs of a right triangle if the hypotenuse is 6, and one angle is twice the other.

51. In a right triangle given c, and $A = nB$; find a and b.

52. In a right triangle the difference between the hypotenuse and the greater leg is equal to the difference between the two legs. Find the angles.

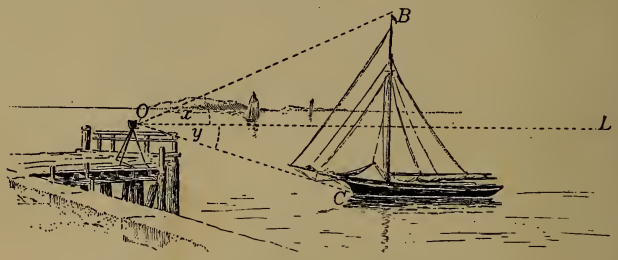


FIG. 20

The *angle of elevation* of an object, or the *angle of depression*, is the angle which a line from the eye to the object makes with a horizontal line in the same vertical plane.

Thus, if the observer is at O (Fig. 20), x is the angle of elevation of B, and y is the angle of depression of C.

53. At a horizontal distance of 120 feet from the foot of a steeple, the angle of elevation of the top was found to be 60° 30′. Find the height of the steeple.

54. From the top of a rock that rises vertically 326 feet out of the water, the angle of depression of a boat was found to be 24°. Find the distance of the boat from the foot of the rock.

55. How far is a monument, in a level plain, from the eye, if the height of the monument is 200 feet and the angle of elevation of the top 3° 30′?

56. A distance AB is measured 96 feet along the bank of a river from a point A opposite a tree C on the other bank. The angle ABC is 21° 14′. Find the breadth of the river.

57. What is the angle of elevation of an inclined plane if it rises 1 foot in a horizontal distance of 40 feet?

58. Find the angle of elevation of the sun when a tower 120 feet high casts a horizontal shadow 70 feet long.

59. How high is a tree that casts a horizontal shadow 80 feet in length when the angle of elevation of the sun is 50°?

60. A ship is sailing due northeast at a rate of 10 miles an hour. Find the rate at which she is moving due north, and also due east.

61. In front of a window 20 feet high is a flower-bed 6 feet wide. How long is a ladder that will just reach from the edge of the bed to the window?

62. A ladder 40 feet long may be so placed that it will reach a window 33 feet high on one side of the street, and by turning it over without moving its foot it will reach a window 21 feet high on the other side. Find the breadth of the street.

63. From the top of a hill the angles of depression of two successive milestones, on a straight level road leading to the hill, are observed to be 5° and 15°. Find the height of the hill.

64. A fort stands on a horizontal plain. The angle of elevation at a certain point on the plain is 30°, and at a point 100 feet nearer the fort it is 45°. How high is the fort?

65. From a certain point on the ground the angles of elevation of the belfry of a church and of the top of the steeple were found to be 40° and 51°, respectively. From a point 300 feet farther off, on a horizontal line, the angle of elevation of the top of the steeple is found to be 33° 45′. Find the distance from the belfry to the top of the steeple.

66. The angle of elevation of the top C of an inaccessible fort observed from a point A is 12°. At a point B, 219 feet from A and on a line AB perpendicular to AC, the angle ABC is 61° 45′. Find the height of the fort.

SECTION XV

THE ISOSCELES TRIANGLE

An isosceles triangle is divided by the perpendicular from the vertex to the base into two *equal right triangles.*

Therefore, an isosceles triangle is determined by any two parts that determine one of these right triangles.

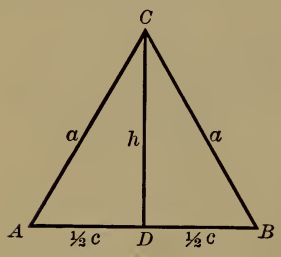


FIG. 21

Let the parts of an isosceles triangle CAB (Fig. 21), among which the altitude CD is to be included, be denoted as follows:

a = one of the equal sides,

c = the base,

h = the altitude,

A = one of the equal angles,

C = the angle at the vertex.

For example: Given a and c; required A, C, h.

1. $\cos A = \frac{\frac{1}{2}c}{a} = \frac{c}{2a}$.

2. $C + 2A = 180°$; $\therefore C = 180° - 2A = 2(90° - A)$.

3. h may be found by any one of the equations:

$$h^2 + \frac{c^2}{4} = a^2;$$

whence $$h = \sqrt{(a + \tfrac{1}{2}c)(a - \tfrac{1}{2}c)}.$$

Also, $$\frac{h}{a} = \sin A, \quad \text{and} \quad \frac{h}{\frac{1}{2}c} = \tan A;$$

whence $$h = a \sin A, \quad \text{and} \quad h = \tfrac{1}{2}c \tan A.$$

When c and h are known, the area can be found by the formula

$$F = \tfrac{1}{2}ch.$$

EXERCISE X

Solve the following isosceles triangles, finding the angles to the nearest second:

1. Given a and A; find C, c, h.
2. Given a and C; find A, c, h.
3. Given c and A; find C, a, h.
4. Given c and C; find A, a, h.
5. Given h and A; find C, a, c.
6. Given h and C; find A, a, c.
7. Given a and h; find A, C, c.
8. Given c and h; find A, C, a.
9. Given $a = 14.3$, $c = 11$; find A, C, h.
10. Given $a = 0.295$, $A = 68° 10'$; find c, h, F.
11. Given $c = 2.352$, $C = 69° 49'$; find a, h, F.
12. Given $h = 7.4847$, $A = 76° 14'$; find a, c, F.
13. Given $a = 6.71$, $h = 6.6$; find A, C, c.
14. Given $c = 9$, $h = 20$; find A, C, a.
15. Given $c = 147$, $F = 2572.5$; find A, C, a, h.
16. Given $h = 16.8$, $F = 43.68$; find A, C, a, c.
17. Find the value of F in terms of a and c.
18. Find the value of F in terms of a and C.
19. Find the value of F in terms of a and A.
20. Find the value of F in terms of h and C.

21. A barn is 40×80 feet, the pitch of the roof is $45°$; find the length of the rafters and the area of the whole roof.

22. In a unit circle what is the length of the chord corresponding to the angle $45°$ at the centre?

23. If the radius of a circle is 30, and the length of a chord is 44, find the angle subtended at the centre.

24. Find the radius of a circle if a chord whose length is 5 subtends at the centre an angle of 133°.

25. What is the angle at the centre of a circle if the corresponding chord is equal to $\frac{2}{3}$ of the radius?

26. Find the area of a circular sector if the radius of the circle is 12, and the angle of the sector is 30°.

SECTION XVI

THE REGULAR POLYGON

Lines drawn from the centre of a regular polygon (Fig. 22) to the vertices are radii of the circumscribed circle; and lines drawn from the centre to the middle points of the sides are radii of the inscribed circle. These lines divide the polygon into *equal right triangles.* Therefore, a regular polygon is determined by a right triangle whose sides are the radius of the circumscribed circle, the radius of the inscribed circle, and half of one side of the polygon.

If the polygon has n sides, the angle of this right triangle at the centre of the polygon is equal to $\frac{1}{2}\left(\frac{360°}{n}\right)$, or $\frac{180°}{n}$; and the triangle may be solved when a side of the polygon or one of the radii is given.

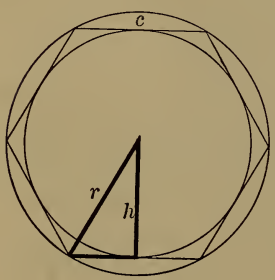


FIG. 22

Let

n = number of sides,
c = length of one side,
r = radius of circumscribed circle,
h = radius of inscribed circle,
p = the perimeter,
F = the area.

Then, by Geometry,

$$F = \tfrac{1}{2}\, hp.$$

EXERCISE XI

Find the remaining parts of a regular polygon, given:

1. $n = 10$, $c = 1$.
2. $n = 18$, $r = 1$.
3. $n = 20$, $r = 20$.
4. $n = 8$, $h = 1$.
5. $n = 11$, $F = 20$.
6. $n = 7$, $F = 7$.

Find the side of:

7. A regular decagon inscribed in a unit circle.

8. A regular decagon circumscribed about a unit circle.

9. If the side of an inscribed regular hexagon is 1, find the side of an inscribed regular dodecagon.

10. Given n and c, and let b denote the side of the inscribed regular polygon having $2n$ sides; find b in terms of n and c.

11. Compute the difference between the areas of a regular octagon and a regular nonagon if the perimeter of each is 16.

12. Compute the difference between the perimeters of a regular pentagon and a regular hexagon if the area of each is 12.

Find the area of:

13. The regular octagon formed by cutting away the corners of a square whose side is 1.

14. A regular pentagon if its diagonals are each equal to 12.

15. A regular polygon of 11 sides inscribed in a circle, if the area of an inscribed regular pentagon is 331.8.

16. A circle inscribed in an equilateral triangle whose perimeter is 20.

17. A regular polygon of 15 sides inscribed in a circle, if the area of a regular inscribed polygon of 16 sides is 100.

18. Find the perimeter of a regular dodecagon circumscribed about a circle the circumference of which is 1.

19. The area of a regular polygon of 25 sides is 40; find the area of the ring comprised between the circumferences of the inscribed and circumscribed circles.

CHAPTER III

GONIOMETRY

SECTION XVII

DEFINITION OF GONIOMETRY

To prepare the way for the solution of the oblique triangle, we now proceed to extend the definitions of the trigonometric functions to angles of all magnitudes, and to deduce certain useful relations of the functions of different angles.

That branch of Trigonometry which treats of trigonometric functions in general, and of their relations, is called **Goniometry.**

SECTION XVIII

POSITIVE AND NEGATIVE QUANTITIES

In measurements it is convenient to mark the distinction between two magnitudes that are measured in *opposite directions,* by calling one of them **positive** and the other **negative.**

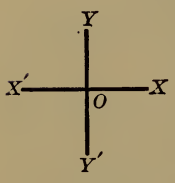


FIG. 23

Thus, if OX (Fig. 23) is considered to be positive, then OX' is considered to be negative; and if OY is considered to be positive, then OY' is considered to be negative.

When this distinction is applied to angles, an angle is considered to be *positive,* if the rotating line that describes it moves counter-clockwise, that is, in the direction opposite

to the hands of a clock, and to be *negative*, if the rotating line moves clockwise, that is, in the same direction as the hands of a clock.

Arcs corresponding to positive angles are considered *positive*, and arcs corresponding to negative angles are considered *negative*.

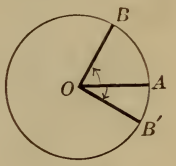


FIG. 24

Thus, the angle AOB (Fig. 24) described by a line rotating about O from OA to OB is positive, and the arc AB is positive; the angle AOB' described by the line rotating about O from OA to OB' is negative, and the arc AB' is negative.

SECTION XIX

CO-ORDINATES OF A POINT IN A PLANE

Let XX' (Fig. 25) be a horizontal line and let YY' be a line perpendicular to XX' at the point O. Then the plane determined by the lines XX' and YY' is divided into four quadrants which are numbered I, II, III, IV.

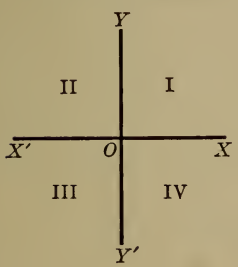


FIG. 25

Any point in the plane is determined by its *distance* and *direction* from each of the perpendiculars XX' and YY'. Its distance from YY', measured on XX', is called the **abscissa** of the point; its distance from XX', measured on YY', is called the **ordinate** of the point.

The abscissa and the ordinate of a point are called the **co-ordinates** of the point; and the lines XX' and YY' are called the **axes of co-ordinates**. XX' is called the **axis of abscissas** or the **axis of x**; YY' is called the **axis of ordinates** or the **axis of y**; and the point O is called the **origin**.

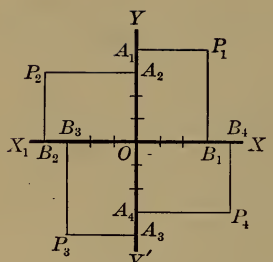


FIG. 26

In Fig. 26 the co-ordinates P_1, P_2, P_3, P_4 are as follows:

The abscissa of P_1 is OB_1,
and the ordinate of P_1 is OA_1;
the abscissa of P_2 is OB_2,
and the ordinate of P_2 is OA_2;
the abscissa of P_3 is OB_3,
and the ordinate of P_3 is OA_3;
the abscissa of P_4 is OB_4,
and the ordinate of P_4 is OA_4.

Abscissas to the *right* of YY' are **positive.**
Abscissas to the *left* of YY' are **negative.**
Ordinates *above* XX' are **positive.**
Ordinates *below* XX' are **negative.**

Therefore,

in Quadrant I,
the abscissa is positive, the ordinate is positive;
in Quadrant II,
the abscissa is negative, the ordinate is positive;
in Quadrant III,
the abscissa is negative, the ordinate is negative;
in Quadrant IV,
the abscissa is positive, the ordinate is negative.

SECTION XX

ANGLES OF ANY MAGNITUDE

If the line OP (Figs. 27–30) is revolved about O from OX as its **initial position** counter-clockwise, as shown by the curved arrows, the line during one revolution will form with OX all angles from 0° to 360°.

Any particular angle is said to be an angle of that quadrant in which its **terminal side** lies.

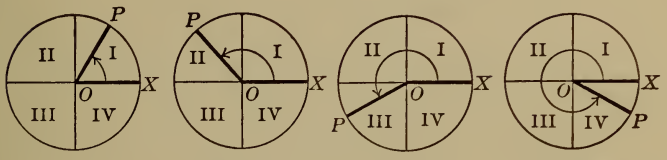


FIG. 27 FIG. 28 FIG. 29 FIG. 30

Angles between 0° and 90° are angles of Quadrant I.
Angles between 90° and 180° are angles of Quadrant II.
Angles between 180° and 270° are angles of Quadrant III.
Angles between 270° and 360° are angles of Quadrant IV.

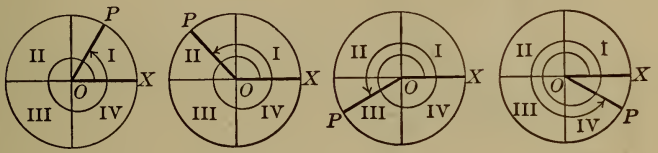


FIG. 31 FIG. 32 FIG. 33 FIG. 34

If the revolving line makes another revolution (Figs. 31–34), it will describe all angles from 360° to 720°; and so on.

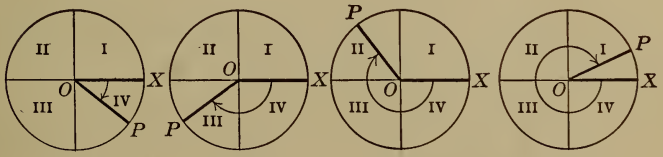


FIG. 35 FIG. 36 FIG. 37 FIG. 38

If the line *OP* is revolved from *OX* clockwise (Figs. 35–38), it will describe all *negative* angles.

Thus we arrive at the conception of an angle of any magnitude, positive or negative.

SECTION XXI

FUNCTIONS OF ANY ANGLE

Figs. 39–42 show the functions in a unit circle drawn for an angle AOP in each quadrant, taken in order. The tangents to the circle are *always* drawn through A and B.

Let the angle AOP formed with OA by the moving radius OP be denoted by x; then, in each quadrant,

$$\sin x = MP, \qquad \tan x = AT, \qquad \sec x = OT,$$
$$\cos x = OM, \qquad \cot x = BS, \qquad \csc x = OS.$$

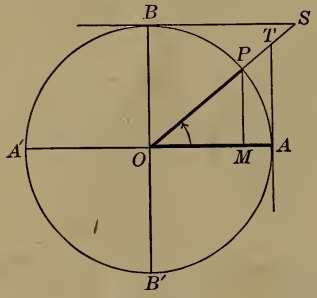


FIG. 39

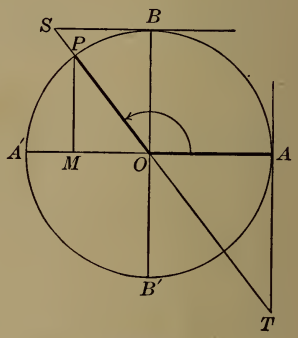


FIG. 40

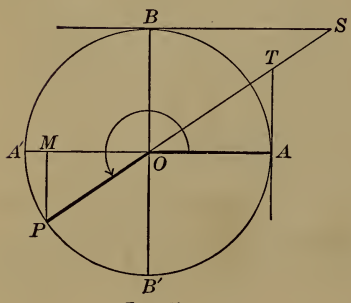


FIG. 41

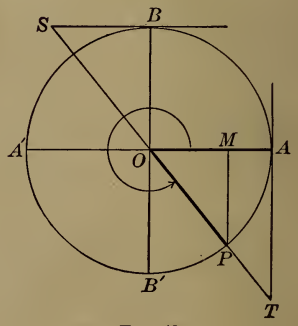


FIG. 42

If the terminal line of any angle x extends through the vertex indefinitely both ways, and if the circumference of a unit circle cuts the terminal line at P, the axis of abscissas at A, and the axis of ordinates at B, then

sin x = *the ordinate of* P ;
cos x = *the abscissa of* P ;
tan x = *the tangent from* A *to meet the terminal line ;*
cot x = *the tangent from* B *to meet the terminal line ;*
sec x = *the segment of the terminal line between the vertex and the tangent ;*
csc x = *the segment of the terminal line between the vertex and the cotangent.*

Sines and tangents extending from the axis of abscissas *upwards* are positive ; *downwards*, negative.

Cosines and cotangents extending from the axis of ordinates *towards the right* are positive ; *towards the left*, negative.

The signs of the secant and cosecant are determined by the signs of the cosine and sine, respectively. Therefore, secants and cosecants extending from the centre, *in the direction of the terminal line*, are considered positive ; *in the opposite direction*, negative. Hence,

Quadrant	I	II	III.	IV
sin and csc	+	+	−	−
cos and sec	+	−	−	+
tan and cot	+	−	+	−

In Quadrant I *all* the functions are positive.
In Quadrant II the *sine and cosecant only* are positive.
In Quadrant III the *tangent and cotangent only* are positive.
In Quadrant IV the *cosine and secant only* are positive.

The signs of all the functions of any quadrant are known when the signs of the sine and cosine are known.

If the sine and cosine have like signs, the tangent and cotangent are positive; if unlike signs, negative. The sine and cosecant have like signs; the cosine and secant have like signs.

SECTION XXII

FUNCTIONS OF A VARIABLE ANGLE

Let the angle AOP (Fig. 43) increase continuously from 0° to 360°. The values of its functions change as follows:

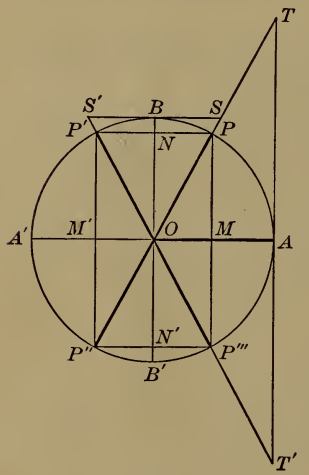


FIG. 43

1. *The Sine.* In the first quadrant the sine MP increases from 0 to 1; in the second it remains positive, and decreases from 1 to 0; in the third it is negative, and increases in absolute value from 0 to 1; in the fourth it is negative, and decreases in absolute value from 1 to 0.

2. *The Cosine.* In the first quadrant the cosine OM decreases from 1 to 0; in the second it becomes negative, and increases in absolute value from 0 to 1; in the third it is negative, and decreases in absolute value from 1 to 0; in the fourth it is positive, and increases from 0 to 1.

3. *The Tangent.* In the first quadrant the tangent AT increases from 0 to ∞; in the second it becomes negative, and decreases in absolute value from ∞ to 0; in the third it is positive, and increases from 0 to ∞; in the fourth it is negative, and decreases in absolute value from ∞ to 0.

4. *The Cotangent.* In the first quadrant, the cotangent BS decreases from ∞ to 0; in the second it is negative, and increases in absolute value from 0 to ∞; in the third and fourth quadrants, it has the same sign, and undergoes the same changes as in the first and second quadrants, respectively.

5. *The Secant.* In the first quadrant, the secant OT increases from 1 to ∞; in the second it is negative, and decreases in absolute value from ∞ to 1; in the third it is negative, and increases in absolute value from 1 to ∞; in the fourth it is positive, and decreases from ∞ to 1.

6. *The Cosecant.* In the first quadrant, the cosecant OS decreases from ∞ to 1; in the second it is positive, and increases from 1 to ∞; in the third it is negative, and decreases in absolute value from ∞ to 1; in the fourth it is negative, and increases in absolute value from 1 to ∞.

The limiting values of the functions are as follows:

	0°	90°	180°	270°	360°
Sine	± 0	$+1$	± 0	-1	± 0
Cosine	$+1$	± 0	-1	± 0	$+1$
Tangent	± 0	$\pm \infty$	± 0	$\pm \infty$	± 0
Cotangent	$\pm \infty$	± 0	$\pm \infty$	± 0	$\pm \infty$
Secant	$+1$	$\pm \infty$	-1	$\pm \infty$	$+1$
Cosecant	$\pm \infty$	$+1$	$\pm \infty$	-1	$\pm \infty$

Sines and cosines vary in value from $+1$ to -1; tangents and cotangents, from $+\infty$ to $-\infty$; secants and cosecants, from $+\infty$ to $+1$, and from -1 to $-\infty$.

In the table given above the double sign $\pm$ is placed before 0 and ∞. From the preceding investigation it appears that the functions *always change sign in passing through* 0 *and* ∞; and the sign $+$ or $-$ prefixed to 0 or ∞ simply shows the direction from which the value is reached.

SECTION XXIII

FUNCTIONS OF ANGLES LARGER THAN 360°

The functions of $360° + x$ are the same in sign and in absolute value as those of x; for the moving radius has the same position in both cases. If n is a positive integer,

The functions of (n × 360° + x) *are the same as those of* x.

For example: The functions of $2200°(6 \times 360° + 40°)$ are equal to the functions of 40°.

SECTION XXIV

EXTENSION OF FORMULAS

The Formulas [1], [2], [3] established for *acute* angles on pp. 13, 14 hold true for *all* angles. Thus, in each quadrant

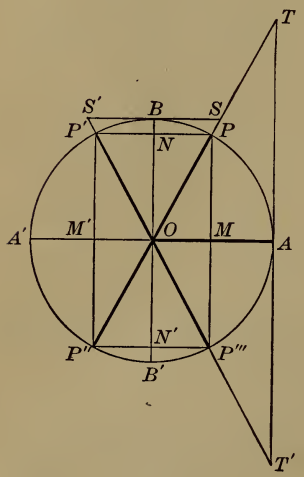


FIG. 44

$$\overline{MP}^2 + \overline{OM}^2 = \overline{OP}^2.$$

Therefore,

$$\sin^2 x + \cos^2 x = 1. \qquad [1]$$

We have in each quadrant from the similar triangles OMP, OAT, OBS the proportions

$AT : OA = MP : OM$,
or $\tan x : 1 = \sin x : \cos x$;

$MP : OP = OB : OS$,
or $\sin x : 1 = 1 : \csc x$;

$OM : OP = OA : OT$,
or $\cos x : 1 = 1 : \sec x$;

$AT : OA = OB : BS$,
or $\tan x : 1 = 1 : \cot x$.

That is,
$$\tan x = \frac{\sin x}{\cos x}. \qquad [2]$$

$$\left.\begin{aligned} \sin x \times \csc x &= 1 \\ \cos x \times \sec x &= 1 \\ \tan x \times \cot x &= 1 \end{aligned}\right\}. \qquad [3]$$

Formulas [1]–[3] enable us, from a given value of one function, to find the *absolute* values of the other five functions, and also the sign of the reciprocal function. But in order to determine the proper signs to be placed before the other four functions, we must know the quadrant to which the angle in question belongs, or the sign of any *one* of these four functions; for, by Sect. XXI, p. 40, it will be seen that *the signs of any two functions that are not reciprocals determine the quadrant to which the angle belongs.*

EXAMPLE. Given $\sin x = +\frac{4}{5}$, and $\tan x$ negative; find the values of the other functions.

Since $\sin x$ is positive, x is an angle in Quadrant I or II; but, since $\tan x$ is negative, Quadrant I is inadmissible.

By [1], $\cos x = \pm\sqrt{1 - \frac{16}{25}} = \pm\frac{3}{5}$.

Since the angle is in Quadrant II, the minus sign must be taken, and we have

$$\cos x = -\frac{3}{5}.$$

By [2] and [3],

$$\tan x = -\frac{4}{3}, \quad \cot x = -\frac{3}{4}, \quad \sec x = -\frac{5}{3}, \quad \csc x = \frac{5}{4}.$$

EXERCISE XII

1. Construct the functions of an angle in Quadrant II. What are their signs?

2. Construct the functions of an angle in Quadrant III. What are their signs?

3. Construct the functions of an angle in Quadrant IV. What are their signs?

4. What are the signs of the functions of the following angles: 340°, 239°, 145°, 400°, 700°, 1200°, 3800°?

5. How many angles less than 360° have the value of the sine equal to $+\frac{5}{7}$, and in what quadrants do they lie?

6. How many values less than 720° can the angle x have if $\cos x = +\frac{2}{3}$, and in what quadrants do they lie?

7. If we take into account only angles less than 180°, how many values can x have if $\sin x = \frac{5}{7}$? if $\cos x = \frac{1}{5}$? if $\cos x = -\frac{4}{5}$? if $\tan x = \frac{2}{3}$? if $\cot x = -7$?

8. Within what limits must the angle x lie if $\cos x = -\frac{2}{9}$? if $\cot x = 4$? if $\sec x = 80$? if $\csc x = -3$? (If $x < 360°$.)

9. In what quadrant does an angle lie if sine and cosine are both negative? if cosine and tangent are both negative? if cotangent is positive and sine negative?

10. Between 0° and 3600° how many angles are there whose sines have the absolute value $\frac{3}{5}$? Of these sines how many are positive and how many negative?

11. In finding $\cos x$ by means of the equation $\cos x = \pm\sqrt{1 - \sin^2 x}$, when must we choose the positive sign and when the negative sign?

12. Given $\cos x = -\sqrt{\frac{1}{2}}$; find the other functions when x is an angle in Quadrant II.

13. Given $\tan x = \sqrt{3}$; find the other functions when x is an angle in Quadrant III.

14. Given $\sec x = +7$, and $\tan x$ negative; find the other functions of x.

15. Given $\cot x = -3$; find all the possible values of the other functions.

16. What functions of an angle of a triangle may be negative? In what case are they negative?

17. Why may cot 360° be considered equal either to $+\infty$ or to $-\infty$?

18. Obtain by means of Formulas [1]–[3] the other functions of the angles, given:

(i) $\tan 90° = \infty$. (iii) $\cot 270° = 0$.

(ii) $\cos 180° = -1$. (iv) $\csc 360° = -\infty$.

19. Find the values of sin 450°, tan 540°, cos 630°, cot 720°, sin 810°, csc 900°.

Compute the values of the following expressions:

20. $a \sin 0° + b \cos 90° - c \tan 180°$.

21. $a \cos 90° - b \tan 180° + c \cot 90°$.

22. $a \sin 90° - b \cos 360° + (a - b) \cos 180°$.

23. $(a^2 - b^2) \cos 360° - 4\, ab \sin 270°$.

SECTION XXV

REDUCTION OF FUNCTIONS TO THE FIRST QUADRANT

In a unit circle (Fig. 45) draw two diameters PR and QS equally inclined to the horizontal diameter AA', or so that the angles AOP, $A'OQ$, $A'OR$, and AOS shall be equal. From the points P, Q, R, S let fall perpendiculars to AA'; the four right triangles thus formed, with a common vertex at O, are equal; because they have equal hypotenuses (radii of the circle) and equal acute angles at O. Therefore, the perpendiculars PM, QN, RN, SM are equal, and are the sines of the angles AOP, AOQ, AOR, AOS, respectively.

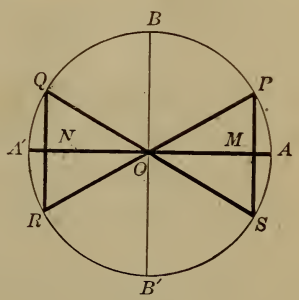


FIG. 45

Therefore, *in absolute value,*

$$\sin AOP = \sin AOQ = \sin AOR = \sin AOS.$$

And from Sect. XXIV, p. 44, it follows that *in absolute value* the cosines of these angles are also equal; and likewise the tangents, the cotangents, the secants, and the cosecants.*

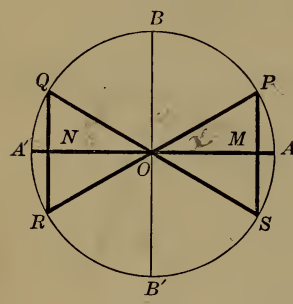


FIG. 46

Hence, *For every acute angle there is an angle in each of the higher quadrants whose functions, in absolute value, are equal to those of this acute angle.*

Let $\angle AOP = x$, $\angle POB = y$; then $x + y = 90°$, and the functions of x are equal to the co-named functions of y (Sect. V, p. 11); and

$$\angle AOQ \text{ (in Quadrant II)} = 180° - x = 90° + y,$$
$$\angle AOR \text{ (in Quadrant III)} = 180° + x = 270° - y,$$
$$\angle AOS \text{ (in Quadrant IV)} = 360° - x = 270° + y.$$

Hence, prefixing the proper sign (Sect. XXI, p. 40), we have:

ANGLE IN QUADRANT II

$\sin(180° - x) = \sin x.$	$\sin(90° + y) = \cos y.$
$\cos(180° - x) = -\cos x.$	$\cos(90° + y) = -\sin y.$
$\tan(180° - x) = -\tan x.$	$\tan(90° + y) = -\cot y.$
$\cot(180° - x) = -\cot x.$	$\cot(90° + y) = -\tan y.$

* In future, secants, cosecants, versed sines, and coversed sines will be disregarded. Secants and cosecants may be found by Formula [3], versed sines and coversed sines by VII and VIII, p. 5, if wanted, but they are seldom used in computations.

Angle in Quadrant III

$$\sin(180° + x) = -\sin x. \qquad \sin(270° - y) = -\cos y.$$
$$\cos(180° + x) = -\cos x. \qquad \cos(270° - y) = -\sin y.$$
$$\tan(180° + x) = \tan x. \qquad \tan(270° - y) = \cot y.$$
$$\cot(180° + x) = \cot x. \qquad \cot(270° - y) = \tan y.$$

Angle in Quadrant IV

$$\sin(360° - x) = -\sin x. \qquad \sin(270° + y) = -\cos y.$$
$$\cos(360° - x) = \cos x. \qquad \cos(270° + y) = \sin y.$$
$$\tan(360° - x) = -\tan x. \qquad \tan(270° + y) = -\cot y.$$
$$\cot(360° - x) = -\cot x. \qquad \cot(270° + y) = -\tan y.$$

Remark. The tangents and cotangents may be found directly from the figure, or by Formula [2].

It is evident, from these formulas,

1. *The functions of all angles can be reduced to the functions of angles in the first quadrant, and therefore to functions of angles not greater than* 45° (Sect. V, p. 11).

2. *If an acute angle is added to or subtracted from* 180° *or* 360°, *the functions of the resulting angle are equal in absolute value to the* **like-named** *functions of the acute angle; but if an acute angle is added to or subtracted from* 90° *or* 270°, *the functions of the resulting angle are equal in absolute value to the* **co-named** *functions of the acute angle.*

3. *A given value of a sine or cosecant determines two supplementary angles, one acute, the other obtuse; a given value of any other function determines only one angle: acute if the value is positive, obtuse if the value is negative.* [See functions of $(180° - x)$.]

SECTION XXVI

FUNCTIONS OF ANGLES THAT DIFFER BY 90°

The general form of two angles whose difference is 90° is x and $90° + x$, and they must lie in adjoining quadrants. The relations between their functions were found in Sect. XXV, p. 48, but only for the case when x is acute. These relations, however, may be shown to hold true for all values of x.

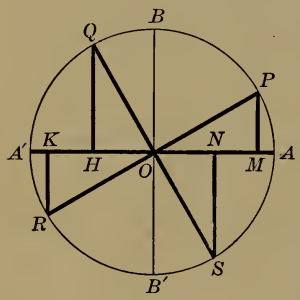


FIG. 47

In a unit circle (Fig. 47) draw two diameters PR and QS perpendicular to each other, and let fall to AA' the perpendiculars PM, QH, RK, and SN. The right triangles OMP, QHO, OKR, and SNO are equal, because they have equal hypotenuses and equal acute angles POM, OQH, ROK, and OSN.

Therefore, $OM = QH = OK = NS$,

and $PM = OH = RK = ON$.

Hence, taking into account the algebraic sign,

$$\sin AOQ = \cos AOP; \quad \sin AOS = \cos AOR;$$
$$\cos AOQ = -\sin AOP; \quad \cos AOS = -\sin AOR;$$
$$\sin AOR = \cos AOQ; \quad \sin (360° + AOP) = \cos AOS;$$
$$\cos AOR = -\sin AOQ; \quad \cos (360° + AOP) = -\sin AOS.$$

In all these equations, if x denotes the angle on the right-hand side, the angle on the left-hand side is $90° + x$.

Therefore, if x is an angle in any one of the four quadrants,

$$\sin (90° + x) = \cos x, \qquad \tan (90° + x) = -\cot x.$$
$$\cos (90° + x) = -\sin x, \qquad \cot (90° + x) = -\tan x.$$

In like manner, it can be shown that all the formulas of Sect. XXV, p. 48, hold true, whatever the values of x and y.

Hence, *In every case the algebraic sign of the function of the resulting angle is the same as when* x *and* y *are both acute.*

SECTION XXVII

FUNCTIONS OF A NEGATIVE ANGLE

If the angle x is generated by the radius moving from the initial position OA to the terminal position OS, it will have the sign $-$, and its terminal side will be identical with its position for the angle $360° - x$. Therefore, the functions of the angle $-x$ are the same as those of the angle $360° - x$; or (Sect. XXV, p. 49),

$$\sin(-x) = -\sin x,$$
$$\cos(-x) = \cos x,$$
$$\tan(-x) = -\tan x,$$
$$\cot(-x) = -\cot x.$$

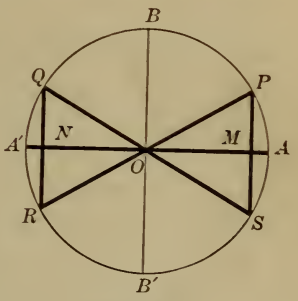


FIG. 48

EXERCISE XIII

1. Express $\sin 250°$ in terms of the functions of an acute angle less than $45°$.

SOLUTION. $\sin 250° = \sin(270° - 20°) = -\cos 20°$.

Express the following functions in terms of the functions of angles less than $45°$:

2. $\sin 172°$.
3. $\cos 100°$.
4. $\tan 125°$.
5. $\cot 91°$.
6. $\sec 110°$.
7. $\csc 157°$.
8. $\sin 204°$.
9. $\cos 359°$.
10. $\tan 300°$.

11. cot 264°. 14. sin 163° 49′. 17. cot 139° 17′.

12. sec 244°. 15. cos 195° 33′. 18. sec 299° 45′.

13. csc 271°. 16. tan 269° 15′. 19. csc 92° 25′.

Express all the functions of the following negative angles in terms of the functions of positive angles less than 45°:

20. $-75°$. 22. $-200°$. 24. $-52° 37′$.

21. $-127°$. 23. $-345°$. 25. $-196° 54′$.

26. Find the functions of 120°.

HINT. $120° = 180° - 60°$, or $90° + 30°$; then apply Sect. XXV, p. 48.

Find the functions of the following angles:

27. 135°. 29. 210°. 31. 240°. 33. $-30°$.

28. 150°. 30. 225°. 32. 300°. 34. $-225°$.

35. Given $\sin x = -\frac{1}{2}\sqrt{2}$, and $\cos x$ negative; find the other functions of x, and the value of x.

36. Given $\cot x = -\sqrt{3}$, and x in Quadrant II; find the other functions of x, and the value of x.

37. Find the functions of 3540°.

38. What angles less than 360° have a sine equal to $-\frac{1}{2}$? a tangent equal to $-\sqrt{3}$?

39. Which of the angles mentioned in Examples 27–34 have a cosine equal to $-\frac{1}{2}\sqrt{2}$? a cotangent equal to $-\sqrt{3}$?

40. What values of x between 0° and 720° will satisfy the equation $\sin x = +\frac{1}{2}$?

41. Find the other angle between 0° and 360° for which the corresponding function (sign included) has the same value as sin 12°, cos 26°, tan 45°, cot 72°, sin 191°, cos 120°, tan 244°, cot 357°.

42. Given $\tan 238° = 1.6$; find $\sin 122°$.

43. Given $\cos 333° = 0.89$; find $\tan 117°$.

Simplify the following expressions:

44. $a \cos(90° - x) + b \cos(90° + x)$.

45. $m \cos(90° - x) \sin(90° - x)$.

46. $(a - b) \tan(90° - x) + (a + b) \cot(90° + x)$.

47. $a^2 + b^2 - 2\,ab \cos(180° - x)$.

48. $\sin(90° + x) \sin(180° + x) + \cos(90° + x) \cos(180° - x)$.

49. $\cos(180° + x) \cos(270° - y) - \sin(180° + x) \sin(270° - y)$.

50. $\tan x + \tan(-y) - \tan(180° - y)$.

51. For what values of x is the expression $\sin x + \cos x$ positive, and for what values negative?

52. Answer the questions of Example 51 for $\sin x - \cos x$.

53. Find the functions of $x - 90°$ in functions of x.

54. Find the functions of $x - 180°$ in functions of x.

SECTION XXVIII

FUNCTIONS OF THE SUM OF TWO ANGLES

In a unit circle (Fig. 49) let the angle $AOB = x$, the angle $BOC = y$; then the angle $AOC = x + y$.

In order to express $\sin(x + y)$ and $\cos(x + y)$ in terms of the sines and cosines of x and y, draw $CF \perp OA$, $CD \perp OB$, $DE \perp OA$, $DG \perp CF$; then $CD = \sin y$, $OD = \cos y$, and the angle $DCG = x$. Also,

$$\sin(x + y) = CF = DE + CG.$$

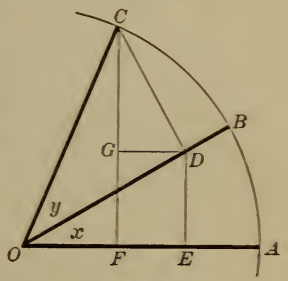


FIG. 49

Now $\frac{DE}{OD} = \sin x$; hence, $DE = \sin x \times OD = \sin x \cos y$.

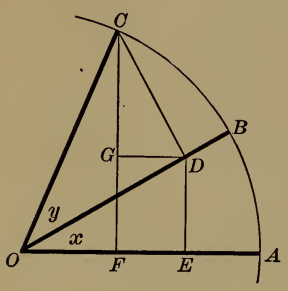


FIG. 50

And $\frac{CG}{CD} = \cos x$; hence,

$CG = \cos x \times CD = \cos x \sin y$.

Therefore,

$$\sin(x+y) = \sin x \cos y + \cos x \sin y. \quad [4]$$

Again,

$\cos(x + y) = OF = OE - DG$.

$\frac{OE}{OD} = \cos x$; hence,

$OE = \cos x \times OD = \cos x \cos y$.

$\frac{DG}{CD} = \sin x$; hence, $DG = \sin x \times CD = \sin x \sin y$.

Therefore, $$\cos(x + y) = \cos x \cos y - \sin x \sin y. \quad [5]$$

In this proof x and y, and also the sum $x + y$, are assumed to be acute angles. If the sum $x + y$ of the acute angles x and y is obtuse, as in Fig. 51, the proof remains, word for word, the same as above, the only difference being that the sign of OF will be negative, as DG is now greater than OE. The above formulas, therefore, hold true for all acute angles x and y.

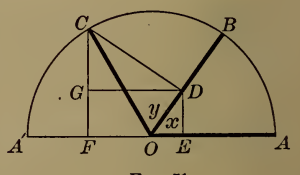


FIG. 51

If these formulas hold true for any two acute angles x and y, they hold true when one of the angles is increased by 90°. Thus, if for x we write $x' = 90° + x$, then, by Sect. XXV, p. 48,

$\sin(x' + y) = \sin(90° + x + y) = \quad \cos(x + y)$,

$\cos(x' + y) = \cos(90° + x + y) = -\sin(x + y)$.

Hence, by [5], $\sin(x' + y) = \cos x \cos y - \sin x \sin y$,

by [4], $\cos(x' + y) = -\sin x \cos y - \cos x \sin y$.

Now, by Sect. XXV, p. 48,

$$\cos x = \sin(90° + x) = \sin x',$$
$$\sin x = -\cos(90° + x) = -\cos x'.$$

Substitute these values of $\cos x$ and $\sin x$, then

$$\sin(x' + y) = \sin x' \cos y + \cos x' \sin y,$$
$$\cos(x' + y) = \cos x' \cos y - \sin x' \sin y.$$

It follows that Formulas [4] and [5] hold true if either angle is repeatedly increased by 90°; therefore they apply to all angles whatever.

By Sect. XXIV, p. 45, Formula [2],

$$\tan(x + y) = \frac{\sin(x + y)}{\cos(x + y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}.$$

If we divide each term of the numerator and denominator of the last fraction by $\cos x \cos y$, we have

$$\tan(x + y) = \frac{\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}}{1 - \frac{\sin x \sin y}{\cos x \cos y}}.$$

That is, $$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}. \qquad [6]$$

Also, $\cot(x + y) = \frac{\cos(x + y)}{\sin(x + y)} = \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y - \cos x \sin y}$.

Divide each term of the numerator and denominator by $\sin x \sin y$, remembering that $\frac{\cos x}{\sin x} = \cot x$ and $\frac{\cos y}{\sin y} = \cot y$; we have

$$\cot(x + y) = \frac{\cot x \cot y - 1}{\cot y + \cot x}. \qquad [7]$$

SECTION XXIX

FUNCTIONS OF THE DIFFERENCE OF TWO ANGLES

In a unit circle (Fig. 52) let the angle $AOB = x$, $COB = y$; then the angle $AOC = x - y$.

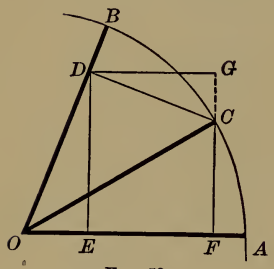


FIG. 52

In order to express $\sin(x - y)$ and $\cos(x - y)$ in terms of the sines and cosines of x and y, draw $CF \perp OA$, $CD \perp OB$, $DE \perp OA$, $DG \perp FC$ prolonged; then $CD = \sin y$, $OD = \cos y$, and the angle $DCG = x$.

Now $\sin(x-y) = CF = DE - CG$.

$$\frac{DE}{OD} = \sin x; \text{ hence, } DE = \sin x \times OD = \sin x \cos y.$$

$$\frac{CG}{CD} = \cos x; \text{ hence, } CG = \cos x \times CD = \cos x \sin y.$$

Therefore, $$\mathbf{\sin(x - y) = \sin x \cos y - \cos x \sin y.} \quad [8]$$

Again, $$\cos(x - y) = OF = OE + DG.$$

$$\frac{OE}{OD} = \cos x; \text{ hence, } OE = \cos x \times OD = \cos x \cos y.$$

$$\frac{DG}{CD} = \sin x; \text{ hence, } DG = \sin x \times CD = \sin x \sin y.$$

Therefore, $$\mathbf{\cos(x - y) = \cos x \cos y + \sin x \sin y.} \quad [9]$$

In this proof, both x and y are assumed to be acute angles; but, whatever be the values of x and y, the same method of proof will always lead to Formulas [8] and [9], when due regard is paid to the algebraic signs.

The general application of these formulas may be at once shown by deducing them from the general formulas established in Sect. XXVIII, p. 54, as follows:

It is obvious that $(x - y) + y = x$. If we apply Formulas [4] and [5] to $(x - y) + y$, then

$$\sin\{(x - y) + y\} \text{ or } \sin x = \sin(x - y)\cos y + \cos(x - y)\sin y,$$
$$\cos\{(x - y) + y\} \text{ or } \cos x = \cos(x - y)\cos y - \sin(x - y)\sin y.$$

Multiply the first equation by $\cos y$, the second by $\sin y$,

$$\sin x \cos y = \sin(x - y)\cos^2 y + \cos(x - y)\sin y \cos y,$$
$$\cos x \sin y = -\sin(x - y)\sin^2 y + \cos(x - y)\sin y \cos y;$$

whence, by subtraction,

$$\sin x \cos y - \cos x \sin y = \sin(x - y)(\sin^2 y + \cos^2 y).$$

But $\sin^2 y + \cos^2 y = 1$ (Sect. XXIV, p. 44).

Therefore, by substitution and transposition,

$$\sin(x - y) = \sin x \cos y - \cos x \sin y.$$

Again, if we multiply the first equation by $\sin y$, the second equation by $\cos y$, and add the results, we obtain, by reducing,

$$\cos(x - y) = \cos x \cos y + \sin x \sin y.$$

Therefore, Formulas [8] and [9], like [4] and [5], from which they have been derived, are universally true.

From [8] and [9], by proceeding as in Sect. XXVIII, p. 55, we obtain

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}. \qquad [10]$$

$$\cot(x - y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}. \qquad [11]$$

Formulas [4]–[11] may be combined as follows:

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},$$
$$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot y \pm \cot x}.$$

SECTION XXX

FUNCTIONS OF TWICE AN ANGLE

If $y = x$, Formulas [4]–[7] become

$$\sin 2x = 2 \sin x \cos x. \qquad [12]$$

$$\cos 2x = \cos^2 x - \sin^2 x. \qquad [13]$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}. \qquad [14]$$

$$\cot 2x = \frac{\cot^2 x - 1}{2 \cot x}. \qquad [15]$$

By these formulas the functions of twice an angle may be found when the functions of the angle are given.

SECTION XXXI

FUNCTIONS OF HALF AN ANGLE

Formula [1] is $\cos^2 x + \sin^2 x = 1.$

Formula [13] is $\cos^2 x - \sin^2 x = \cos 2x.$

Subtract, $2 \sin^2 x = 1 - \cos 2x.$

Add, $2 \cos^2 x = 1 + \cos 2x.$

Whence,

$$\sin x = \pm \sqrt{\frac{1 - \cos 2x}{2}}, \quad \cos x = \pm \sqrt{\frac{1 + \cos 2x}{2}}.$$

These values, if z is put for $2x$, and hence $\frac{1}{2}z$ for x, become

$$\sin \tfrac{1}{2} z = \pm \sqrt{\frac{1 - \cos z}{2}}. \qquad [16]$$

$$\cos \tfrac{1}{2} z = \pm \sqrt{\frac{1 + \cos z}{2}}. \qquad [17]$$

Hence, by division (Sect. XXIV, p. 45),

$$\tan \tfrac{1}{2} z = \pm \sqrt{\frac{1 - \cos z}{1 + \cos z}}. \qquad [18]$$

$$\cot \tfrac{1}{2} z = \pm \sqrt{\frac{1 + \cos z}{1 - \cos z}}. \qquad [19]$$

By these formulas the functions of half an angle may be computed when the cosine of the entire angle is given.

The proper sign to be placed before the root in each case depends on the quadrant in which the angle $\frac{1}{2} z$ lies (Sect. XXII, p. 42).

Let the student show from Formula [18] that

$$\tan \tfrac{1}{2} B = \sqrt{\frac{c - a}{c + a}}. \quad \text{(See p. 23, Note.)}$$

SECTION XXXII

SUMS AND DIFFERENCES OF FUNCTIONS

From [4], [5], [8], and [9], by addition and subtraction,

$$\begin{aligned}
\sin(x + y) + \sin(x - y) &= 2 \sin x \cos y, \\
\sin(x + y) - \sin(x - y) &= 2 \cos x \sin y, \\
\cos(x + y) + \cos(x - y) &= 2 \cos x \cos y, \\
\cos(x + y) - \cos(x - y) &= -2 \sin x \sin y;
\end{aligned}$$

or, by making $x + y = A$, and $x - y = B$, and, therefore, $x = \frac{1}{2}(A + B)$, and $y = \frac{1}{2}(A - B)$,

$$\sin A + \sin B = 2 \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B). \qquad [20]$$

$$\sin A - \sin B = 2 \cos \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B). \qquad [21]$$

$$\cos A + \cos B = 2 \cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B). \qquad [22]$$

$$\cos A - \cos B = -2 \sin \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B). \qquad [23]$$

From [20] and [21], by division, we obtain

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \tan \tfrac{1}{2}(A + B) \cot \tfrac{1}{2}(A - B);$$

or, since $$\cot \tfrac{1}{2}(A - B) = \frac{1}{\tan \tfrac{1}{2}(A - B)},$$

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \tfrac{1}{2}(A + B)}{\tan \tfrac{1}{2}(A - B)}. \qquad [24]$$

EXERCISE XIV

1. Find the value of $\sin(x + y)$ and $\cos(x + y)$ when $\sin x = \frac{3}{5}$, $\cos x = \frac{4}{5}$, $\sin y = \frac{5}{13}$, $\cos y = \frac{12}{13}$.

2. Find $\sin(90° - y)$ and $\cos(90° - y)$ by making $x = 90°$ in Formulas [8] and [9].

Find, by Formulas [4]–[11], the first four functions of:

3. $90° + y$.
4. $180° - y$.
5. $180° + y$.
6. $270° - y$.
7. $270° + y$.
8. $360° - y$.
9. $360° + y$.
10. $x - 90°$.
11. $x - 180°$.
12. $x - 270°$.
13. $-y$.
14. $45° - y$.
15. $45° + y$.
16. $30° + y$.
17. $60° - y$.
18. Find $\sin 3x$ in terms of $\sin x$.
19. Find $\cos 3x$ in terms of $\cos x$.
20. Given $\tan \frac{1}{2}x = 1$; find $\cos x$.
21. Given $\cot \frac{1}{2}x = \sqrt{3}$; find $\sin x$.
22. Given $\sin x = 0.2$; find $\sin \frac{1}{2}x$ and $\cos \frac{1}{2}x$.
23. Given $\cos x = 0.5$; find $\cos 2x$ and $\tan 2x$.
24. Given $\tan 45° = 1$; find the functions of $22° 30'$.
25. Given $\sin 30° = 0.5$; find the functions of $15°$.
26. Prove that $\tan 18° = \dfrac{\sin 33° + \sin 3°}{\cos 33° + \cos 3°}$.

Prove the following formulas:

27. $\sin 2x = \dfrac{2\tan x}{1+\tan^2 x}$.

28. $\cos 2x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$.

29. $\tan \frac{1}{2}x = \dfrac{\sin x}{1+\cos x}$.

30. $\cot \frac{1}{2}x = \dfrac{\sin x}{1-\cos x}$.

31. $\sin \frac{1}{2}x \pm \cos \frac{1}{2}x = \sqrt{1 \pm \sin x}$.

32. $\dfrac{\tan x \pm \tan y}{\cot x \pm \cot y} = \pm \tan x \tan y$.

33. $\tan (45° - x) = \dfrac{1-\tan x}{1+\tan x}$.

If A, B, C are the angles of a triangle, prove that:

34. $\sin A + \sin B + \sin C = 4 \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C$.

35. $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C$.

36. $\tan A + \tan B + \tan C = \tan A \times \tan B \times \tan C$.

37. $\cot \frac{1}{2}A + \cot \frac{1}{2}B + \cot \frac{1}{2}C = \cot \frac{1}{2}A \times \cot \frac{1}{2}B \times \cot \frac{1}{2}C$.

Change to a form more convenient for logarithmic computation:

38. $\cot x + \tan x$.

39. $\cot x - \tan x$.

40. $\cot x + \tan y$.

41. $\cot x - \tan y$.

42. $\dfrac{1-\cos 2x}{1+\cos 2x}$.

43. $1 + \tan x \tan y$.

44. $1 - \tan x \tan y$.

45. $\cot x \cot y + 1$.

46. $\cot x \cot y - 1$.

47. $\dfrac{\tan x + \tan y}{\cot x + \cot y}$.

SECTION XXXIII

ANTI-TRIGONOMETRIC FUNCTIONS

If y is any trigonometric function of an angle x, then x is said to be the corresponding *anti-trigonometric* function of y.

Thus, if $y = \sin x$, x is the *anti-sine* or *inverse sine* of y.

The anti-trigonometric functions of y are written

$$\sin^{-1}y, \quad \tan^{-1}y, \quad \sec^{-1}y, \quad \text{vers}^{-1}y,$$
$$\cos^{-1}y, \quad \cot^{-1}y, \quad \csc^{-1}y, \quad \text{covers}^{-1}y.$$

These are read, the angle whose sine is y, and so on.

For example, $\sin 30° = \frac{1}{2}$; hence, $30° = \sin^{-1}\frac{1}{2}$. Similarly, $90° = \cos^{-1}0 = \sin^{-1}1$, and $45° = \tan^{-1}1 = \sin^{-1}\frac{1}{2}\sqrt{2}$, etc.

The symbol -1 must not be confused with the exponent -1. Thus, $\sin^{-1}x$ is a very different expression from $\frac{1}{\sin x}$, which would be written $(\sin x)^{-1}$. On the continent of Europe mathematical writers employ the notation *arc sin*, *arc cos*, etc., for $\sin^{-1}$, $\cos^{-1}$, etc.

There is an important difference between the trigonometric and the anti-trigonometric functions. When an angle is given, its functions are all completely determined; but when one of the functions is given, the angle may have any one of an indefinite number of values. Thus, if $\sin y = \frac{1}{2}$, y may be 30°, or 150°, or either of these increased or diminished by any integral multiple of 360° or 2π, but cannot take any other values. Accordingly, $\sin^{-1}\frac{1}{2} = 30° \pm 2n\pi$, or $150° \pm 2n\pi$, where n is any positive integer. Similarly, $\tan^{-1}1 = 45° \pm 2n\pi$ or $225° \pm 2n\pi$; *i.e.*, $\tan^{-1}1 = 45° \pm n\pi$.

Since one of the angles whose sine is x and one of the angles whose cosine is x together make 90°, and since similar relations hold for the tangent and cotangent, for the secant and cosecant, and for the versed sine and coversed sine, we have

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, \qquad \sec^{-1}x + \csc^{-1}x = \frac{\pi}{2},$$
$$\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, \quad \text{vers}^{-1}x + \text{covers}^{-1}x = \frac{\pi}{2},$$

where it must be understood that each equation is true only for a particular choice of the various possible values of the functions. Thus, if x is positive, and if the angles are always taken in the first quadrant, the equations are correct.

EXERCISE XV

1. Find all the values of the following functions:

$\sin^{-1}\frac{1}{2}\sqrt{3}$, $\tan^{-1}\frac{1}{3}\sqrt{3}$, $\operatorname{vers}^{-1}\frac{1}{2}$, $\cos^{-1}(-\frac{1}{2}\sqrt{2})$,
$\csc^{-1}\sqrt{2}$, $\tan^{-1}\infty$, $\sec^{-1}2$, $\cos^{-1}(-\frac{1}{2}\sqrt{3})$.

2. Prove that

$$\sin^{-1}(-x) = -\sin^{-1}x;\ \cos^{-1}(-x) = \pi - \cos^{-1}x.$$

3. If $\sin^{-1}x + \sin^{-1}y = \pi$, prove that $x = y$.

4. If $y = \sin^{-1}\frac{1}{3}$, find $\tan y$.

5. Prove that $\cos(\sin^{-1}x) = \sqrt{1 - x^2}$.

6. Prove that $\cos(2\sin^{-1}x) = 1 - 2x^2$.

7. Prove that $\tan(\tan^{-1}x + \tan^{-1}y) = \dfrac{x + y}{1 - xy}$.

8. If $x = \sqrt{\frac{1}{2}}$, find all the values of $\sin^{-1}x + \cos^{-1}x$.

9. Prove that $\tan^{-1}\left(\dfrac{x}{\sqrt{1 - x^2}}\right) = \sin^{-1}x$.

10. Find the value of $\sin(\tan^{-1}{}_{12}^{5})$.

11. Find the value of $\cot(2\sin^{-1}\frac{3}{5})$.

12. Find the value of $\sin(\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3})$.

13. If $\sin^{-1}x = 2\cos^{-1}x$, find x.

14. Prove that $\tan(2\tan^{-1}x) = \dfrac{2x}{1 - x^2}$.

15. Prove that $\sin(2\tan^{-1}x) = \dfrac{2x}{1 + x^2}$.

CHAPTER IV

THE OBLIQUE TRIANGLE

SECTION XXXIV

LAW OF SINES

Let A, B, C denote the angles of a triangle ABC (Figs. 53 and 54), and a, b, c, respectively, the lengths of the opposite sides.

Draw $CD \perp AB$, and meeting AB (Fig. 53) or AB produced (Fig. 54) at D. Let $CD = h$.

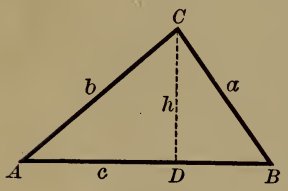


FIG. 53

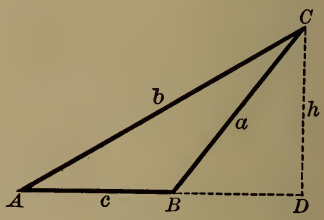


FIG. 54

In either figure, $$\frac{h}{b} = \sin A.$$

In Fig. 53, $$\frac{h}{a} = \sin B.$$

In Fig. 54, $$\frac{h}{a} = \sin (180° - B) = \sin B.$$

Therefore, whether h lies within or without the triangle, we obtain, by division,

$$\frac{a}{b} = \frac{\sin A}{\sin B}. \qquad [25]$$

By drawing perpendiculars from the vertices A and B to the opposite sides, we may obtain, in the same way,

$$\frac{b}{c} = \frac{\sin B}{\sin C}, \qquad \frac{a}{c} = \frac{\sin A}{\sin C}.$$

Hence the Law of Sines:

The sides of a triangle are proportional to the sines of the opposite angles.

If we regard these three equations as proportions, and take them by alternation, it is evident that they may be written in the symmetrical form

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Each of these equal ratios has a simple geometrical meaning which will appear if the Law of Sines is proved as follows:

Circumscribe a circle about the triangle ABC (Fig. 55), and draw the radii OB, OC. Let R denote the radius. Draw $OM \perp BC$. By Geometry, the angle $BOC = 2A$; hence, the angle $BOM = A$, then

$$BM = R \sin BOM = R \sin A.$$

$$\therefore BC \text{ or } a = 2R \sin A.$$

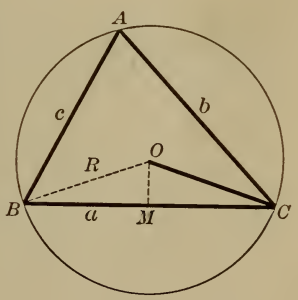


FIG. 55

In like manner,

$$b = 2R \sin B, \text{ and } c = 2R \sin C.$$

Whence we obtain

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

That is: *The ratio of any side of a triangle to the sine of the opposite angle is numerically equal to the diameter of the circumscribed circle.*

SECTION XXXV

LAW OF COSINES

This law gives the value of one side of a triangle in terms of the other two sides and the angle included between them.

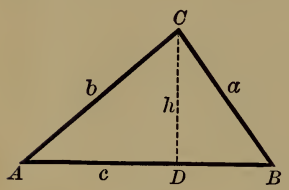


FIG. 56

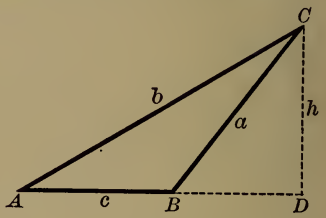


FIG. 57

In Figs. 56 and 57, $a^2 = h^2 + \overline{BD}^2$.

In Fig. 56, $BD = c - AD$.

In Fig. 57, $BD = AD - c$;

In either case, $\overline{BD}^2 = \overline{AD}^2 - 2c \times AD + c^2$.

Therefore, in all cases, $a^2 = h^2 + \overline{AD}^2 + c^2 - 2c \times AD$.

Now, $h^2 + \overline{AD}^2 = b^2$,

and $AD = b \cos A$.

Therefore, $$\mathbf{a^2 = b^2 + c^2 - 2\,bc \cos A.} \qquad [26]$$

In like manner it may be proved that

$$b^2 = a^2 + c^2 - 2\,ac \cos B,$$

$$c^2 = a^2 + b^2 - 2\,ab \cos C.$$

The three formulas have precisely the same form, and the Law of Cosines may be stated as follows:

The square of any side of a triangle is equal to the sum of the squares of the other two sides diminished by twice their product into the cosine of the included angle.

SECTION XXXVI

LAW OF TANGENTS

By Sect. XXXIV, p. 64, $a : b = \sin A : \sin B$;
whence, by the Theory of Proportion,

$$\frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B}.$$

But by [24], p. 60,

$$\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{\tan \tfrac{1}{2}(A-B)}{\tan \tfrac{1}{2}(A+B)}.$$

Therefore,

$$\frac{\mathbf{a-b}}{\mathbf{a+b}} = \frac{\mathbf{\tan \tfrac{1}{2}(A-B)}}{\mathbf{\tan \tfrac{1}{2}(A+B)}}. \qquad [27]$$

By merely changing the letters,

$$\frac{a-c}{a+c} = \frac{\tan \tfrac{1}{2}(A-C)}{\tan \tfrac{1}{2}(A+C)}, \qquad \frac{b-c}{b+c} = \frac{\tan \tfrac{1}{2}(B-C)}{\tan \tfrac{1}{2}(B+C)}.$$

Hence the Law of Tangents:

The difference of two sides of a triangle is to their sum as the tangent of half the difference of the opposite angles is to the tangent of half their sum.

NOTE. If in [27] $b > a$, then $B > A$. The formula is still true, but to avoid negative numbers the formula in this case should be written

$$\frac{b-a}{b+a} = \frac{\tan \tfrac{1}{2}(B-A)}{\tan \tfrac{1}{2}(B+A)}.$$

EXERCISE XVI

1. What do the formulas of Sect. XXXIV, p. 64, become when one of the angles is a right angle?

2. Prove by means of the Law of Sines that the bisector of an angle of a triangle divides the opposite side into parts proportional to the adjacent sides.

3. What does Formula [26] become when $A = 90°$? when $A = 0°$? when $A = 180°$? What does the triangle become in each of these cases?

NOTE. The case when $A = 90°$ explains why the theorem of Sect. XXXV, p. 66, is sometimes called the *Generalized Theorem of Pythagoras.*

4. Prove (Figs. 56 and 57) that whether the angle B is acute or obtuse $c = a \cos B + b \cos A$. What are the two symmetrical formulas obtained by changing the letters? What does the formula become when $B = 90°$?

5. From the three following equations (found in the last example) prove the theorem of Sect. XXXV, p. 66:

$$c = a \cos B + b \cos A,$$
$$b = a \cos C + c \cos A,$$
$$a = b \cos C + c \cos B.$$

HINT. Multiply the first equation by c, the second by b, the third by a; then from the first subtract the *sum* of the second and third.

6. In Formula [27] what is the maximum value of $\frac{1}{2}(A-B)$?

7. Find the form to which Formula [27] reduces, and describe the nature of the triangle, when

(i) $C = 90°$; (ii) $A - B = 90°$, and $B = C$.

SECTION XXXVII

THE GIVEN PARTS

The formulas established in Sects. XXXIV–XXXVI, pp. 64–67, together with the equation $A + B + C = 180°$, are sufficient for solving every case of an oblique triangle. The three parts that determine an oblique triangle may be:

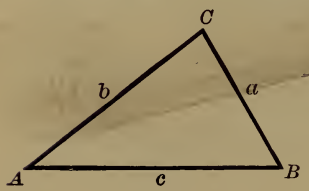


FIG. 58

I. One side and two angles;
II. Two sides and the angle opposite one of these sides;
III. Two sides and the included angle;
IV. The three sides.

SECTION XXXVIII

SOLUTION OF AN OBLIQUE TRIANGLE

Case I

Given one side a *and two angles* A *and* B; *find the remaining parts* C, b, *and* c.

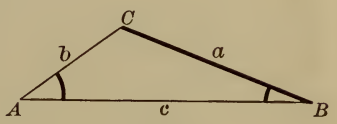


Fig. 59

1. $C = 180° - (A + B)$.
2. $\frac{b}{a} = \frac{\sin B}{\sin A}$; $\therefore b = \frac{a \sin B}{\sin A} = \frac{a}{\sin A} \times \sin B$.
3. $\frac{c}{a} = \frac{\sin C}{\sin A}$; $\therefore c = \frac{a \sin C}{\sin A} = \frac{a}{\sin A} \times \sin C$.

Example. $a = 24.31$, $A = 45° 18'$, $B = 22° 11'$.

The work may be arranged as follows:

$a = 24.31$	$\log a = 1.38578$	$= 1.38578$
$A = 45° 18'$	$\operatorname{colog} \sin A = 0.14825$	$= 0.14825$
$B = 22° 11'$	$\log \sin B = 9.57700$	$\log \sin C = 9.96556$
$A + B = 67° 29'$	$\log b = 1.11103$	$\log c = \overline{1}.49959$
$C = 112° 31'$	$b = 12.913$	$c = 31.593$

Note. When -10 is omitted after a logarithm or cologarithm, it must be remembered that the log or colog is 10 too large.

EXERCISE XVII

1. Given $a = 500$, $A = 10° 12'$, $B = 46° 36'$;
   find $C = 123° 12'$, $b = 2051.5$, $c = 2362.6$.
2. Given $a = 795$, $A = 79° 59'$, $B = 44° 41'$;
   find $C = 55° 20'$, $b = 567.69$, $c = 663.99$.

3. Given $a = 804$, $A = 99° 55'$, $B = 45° 1'$;
find $C = 35° 4'$, $b = 577.31$, $c = 468.93$.

4. Given $a = 820$, $A = 12° 49'$, $B = 141° 59'$;
find $C = 25° 12'$, $b = 2276.6$, $c = 1573.9$.

5. Given $c = 1005$, $A = 78° 19'$, $B = 54° 27'$;
find $C = 47° 14'$, $a = 1340.6$, $b = 1113.8$.

6. Given $b = 13.57$, $B = 13° 57'$, $C = 57° 13'$;
find $A = 108° 50'$, $a = 53.276$, $c = 47.324$.

7. Given $a = 6412$, $A = 70° 55'$, $C = 52° 9'$;
find $B = 56° 56'$, $b = 5685.9$, $c = 5357.5$.

8. Given $b = 999$, $A = 37° 58'$, $C = 65° 2'$;
find $B = 77°$, $a = 630.77$, $c = 929.48$.

9. In order to determine the distance of a hostile fort A from a place B, a line BC and the angles ABC and BCA were measured and found to be 322.55 yards, 60° 34′, and 56° 10′, respectively. Find the distance AB.

10. The angles B and C of a triangle ABC are 50° 30′ and 122° 9′, respectively, and BC is 9 miles. Find AB and AC.

11. Two observers 5 miles apart on a plain, and facing each other, find that the angles of elevation of a balloon in the same vertical plane with themselves are 55° and 58°, respectively. Find the distance from the balloon to each observer, and also the height of the balloon above the plain.

12. In a parallelogram given a diagonal d and the angles x and y which this diagonal makes with the sides; find the sides. Find the sides if $d = 11.237$, $x = 19° 1'$, and $y = 42° 54'$.

13. A lighthouse was observed from a ship to bear N. 34° E.; after the ship sailed due south 3 miles it bore N. 23° E. Find the distance from the lighthouse to the ship in each position.

NOTE. The phrase *to bear N. 34° E.* means that the line of sight to the lighthouse is in the northeast quarter of the horizon and makes, with a line due north, an angle of 34°.

14. In a trapezoid given the parallel sides a and b, and the angles x and y at the ends of one of the parallel sides; find the non-parallel sides. Compute the results when $a = 15$, $b = 7$, $x = 70°$, $y = 40°$.

Solve the following examples without using logarithms:

15. Given $b = 7.07107$, $A = 30°$, $C = 105°$; find a and c.

16. Given $c = 9.562$, $A = 45°$, $B = 60°$; find a and b.

17. The base of a triangle is 600 feet and the angles at the base are 30° and 120°. Find the other sides and the altitude.

18. Two angles of a triangle are, the one 20°, the other 40°. Find the ratio of the opposite sides.

19. The angles of a triangle are as 5 : 10 : 21, and the side opposite the smallest angle is 3. Find the other sides.

20. Given one side of a triangle equal to 27, the adjacent angles equal each to 30°; find the radius of the circumscribed circle. (See Sect. XXXIV, p. 65.)

SECTION XXXIX

Case II

Given two sides a *and* b *and the angle* A *opposite the side* a; *find the remaining parts* B, C, c.

This case, like the preceding case, is solved by means of the Law of Sines.

Since $\dfrac{\sin B}{\sin A} = \dfrac{b}{a}$, therefore $\sin B = \dfrac{b \sin A}{a}$;

$$C = 180° - (A + B).$$

And since $\dfrac{c}{a} = \dfrac{\sin C}{\sin A}$, therefore $c = \dfrac{a \sin C}{\sin A}$.

When an angle is determined by its sine it admits of two values which are supplements of each other (Sect. XXV, p. 48); hence, either value of B may be taken unless excluded by the conditions of the problem.

If $a > b$, then by Geometry $A > B$, and B must be acute whatever be the value of A; for a triangle can have only one obtuse angle. Hence, there is *one, and only one, triangle* that will satisfy the given conditions.

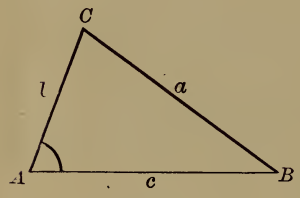


FIG. 60

If $a = b$, then by Geometry $A = B$; both A and B must be acute, and *the required triangle is isosceles.*

If $a < b$, then by Geometry $A < B$, and A must be acute in order that the triangle may be possible. If A is acute, it is evident from Fig. 61, where $\angle BAC = A$, $AC = b$, $CB = CB' = a$, that *the two triangles* ACB *and* ACB′ *will satisfy the given conditions*, provided a is greater than the perpendicular CP; that is, provided a is greater than $b \sin A$ (Sect. XI, p. 20). The angles ABC and $AB'C$ are supplementary (since $\angle ABC = \angle BB'C$); they are, in fact, the supplementary angles obtained from the formula

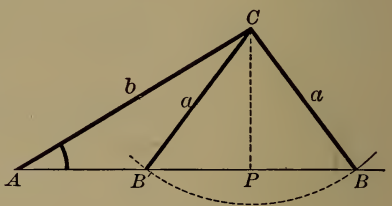


FIG. 61

$$\sin B = \frac{b \sin A}{a}.$$

If, however, $a = b \sin A = CP$ (Fig. 61), then $\sin B = 1$, $B = 90°$, and *the triangle required is a right triangle.*

If $a < b \sin A$, that is, $< CP$, then $\sin B > 1$, and *the triangle is impossible.*

These results, for convenience, may be thus stated:

Two solutions; if A is acute and the value of a lies between b and $b \sin A$.

No solution; if A is acute and $a < b \sin A$;

or if A is obtuse and $a < b$, or $a = b$.

One solution; in all other cases.

The number of solutions can often be determined by inspection. In case of doubt, find the value of $b \sin A$.

Or we may proceed to compute $\log \sin B$. If $\log \sin B = 0$, the triangle required is a right triangle. If $\log \sin B > 0$, the triangle is impossible. If $\log \sin B < 0$, there is *one solution* when $a > b$; there are *two solutions* when $a < b$.

When there are two solutions, let B', C', c', denote the unknown parts of the second triangle; then,

$$B' = 180° - B, \quad C' = 180° - (A + B') = B - A,$$

$$c' = \frac{a \sin C'}{\sin A}.$$

EXAMPLE 1. Given $a = 16$, $b = 20$, $A = 106°$; find the remaining parts.

In this case $a < b$ and $A > 90°$; therefore, the triangle is impossible.

EXAMPLE 2. Given $a = 36$, $b = 80$, $A = 30°$; find the remaining parts.

Here we have $b \sin A = 80 \times \frac{1}{2} = 40$; so that $a < b \sin A$ and the triangle is impossible.

EXAMPLE 3. Given $a = 72{,}630$, $b = 117{,}480$, $A = 80° 0' 50''$; find B, C, c.

$a = 72{,}630$	$\operatorname{colog} a = 5.13888$	Here $\log \sin B > 0$.
$b = 117{,}480$	$\log b = 5.06997$	$\therefore$ *no solution.*
$A = 80° 0' 50''$	$\log \sin A = \underline{9.99337}$	
	$\log \sin B = 0.20222$	

EXAMPLE 4. Given $a = 13.2$, $b = 15.7$, $A = 57° 13' 15''$; find B, C, c.

$a = 13.2$	$\operatorname{colog} a = 8.87943$	$c = b \cos A$
$b = 15.7$	$\log b = 1.19590$	$\log b = 1.19590$
$A = 57° 13' 15''$	$\log \sin A = 9.92467$	$\log \cos A = \overline{9.73352}$
Here $\log \sin B = 0$.	$\log \sin B = \overline{0.00000}$	$\log c = \overline{0.92942}$
$\therefore$ a *right* triangle.	$B = 90°$	$c = 8.5$
	$\therefore C = 32° 46' 45''$	

EXAMPLE 5. Given $a = 767$, $b = 242$, $A = 36° 53' 2''$; find B, C, c.

$a = 767$	$\operatorname{colog} a = 7.11520$	$\log a = 2.88480$
$b = 242$	$\log b = 2.38382$	$\log \sin C = 9.86970$
$A = 36° 53' 2''$	$\log \sin A = \overline{9.77830}$	$\operatorname{colog} \sin A = \overline{0.22170}$
Here $a > b$,	$\log \sin B = \overline{9.27732}$	$\log c = \overline{2.97620}$
and $\log \sin B < 0$.	$B = 10° 54' 58''$	$c = 946.68$
$\therefore$ *one solution.*	$\therefore C = 132° 12' 0''$	

EXAMPLE 6. Given $a = 177.01$, $b = 216.45$, $A = 35° 36' 20''$; find the other parts.

$a = 177.01$	$\operatorname{colog} a = 7.75200$	$\log a = 2.24800$	2.24800
$b = 216.45$	$\log b = 2.33536$	$\log \sin C = 9.99462$	9.23035
$A = 35° 36' 20''$	$\log \sin A = \overline{9.76507}$	$\operatorname{colog} \sin A = \overline{0.23493}$	0.23493
Here $a < b$,	$\log \sin B = \overline{9.85243}$	$\log c = 2.47755$	1.71328
and $\log \sin B < 0$.	$B = 45° 23' 28''$	$c = 300.29$ or 51.675	
$\therefore$ *two solutions.*	or $134° 36' 32''$		
	$\therefore C = 99° 0' 12''$		
	or $9° 47' 8''$		

EXERCISE XVIII

1. Find the number of solutions of the following:

(i) $a = 80$, $b = 100$, $A = 30°$.

(ii) $a = 50$, $b = 100$, $A = 30°$.

(iii) $a = 40$, $b = 100$, $A = 30°$.

(iv) $a = 13.4$, $b = 11.46$, $A = 77° 20'$.

(v) $a = 70$, $b = 75$, $A = 60°$.
(vi) $a = 134.16$, $b = 84.54$, $B = 52° 9' 11''$.
(vii) $a = 200$, $b = 100$, $A = 30°$.

2. Given $a = 840$, $b = 485$, $A = 21° 31'$;
find $B = 12° 13' 34''$, $C = 146° 15' 26''$, $c = 1272.1$.

3. Given $a = 9.399$, $b = 9.197$, $A = 120° 35'$;
find $B = 57° 23' 40''$, $C = 2° 1' 20''$, $c = 0.38525$.

4. Given $a = 91.06$, $b = 77.04$, $A = 51° 9' 6''$;
find $B = 41° 13'$, $C = 87° 37' 54''$, $c = 116.82$.

5. Given $a = 55.55$, $b = 66.66$, $B = 77° 44' 40''$;
find $A = 54° 31' 13''$, $C = 47° 44' 7''$, $c = 50.481$.

6. Given $a = 309$, $b = 360$, $A = 21° 14' 25''$;
find $B = 24° 57' 54''$, $C = 133° 47' 41''$, $c = 615.67$,
$B' = 155° 2' 6''$, $C' = 3° 43' 29''$, $c' = 55.41$.

7. Given $a = 8.716$, $b = 9.787$, $A = 38° 14' 12''$;
find $B = 44° 1' 28''$, $C = 97° 44' 20''$, $c = 13.954$,
$B' = 135° 58' 32''$, $C' = 5° 47' 16''$, $c' = 1.4202$.

8. Given $a = 4.4$, $b = 5.21$, $A = 57° 37' 17''$;
find $B = 90°$, $C = 32° 22' 43''$, $c = 2.7901$.

9. Given $a = 34$, $b = 22$, $B = 30° 20'$;
find $A = 51° 18' 27''$, $C = 98° 21' 33''$, $c = 43.098$,
$A' = 128° 41' 33''$, $C' = 20° 58' 27''$, $c' = 15.593$.

10. Given $b = 19$, $c = 18$, $C = 15° 49'$;
find $B = 16° 43' 13''$, $A = 147° 27' 47''$, $a = 35.519$,
$B' = 163° 16' 47''$, $A = 0° 54' 13''$, $a' = 1.0415$.

11. Given $a = 75$, $b = 29$, $B = 16° 15' 36''$; find the difference between the areas of the two corresponding triangles.

12. Given in a parallelogram the side a, a diagonal d, and the angle A made by the two diagonals; find the other diagonal. Special case: $a = 35$, $d = 63$, $A = 21° 36' 30''$.

SECTION XL

Case III

Given two sides a *and* b *and the included angle* C; *find the remaining parts* A, B, *and* c.

Solution I. The angles A and B may both be found by means of Formula [27], Sect. XXXVI, p. 67, which may be written

$$\tan \tfrac{1}{2}(A - B) = \frac{a - b}{a + b} \times \tan \tfrac{1}{2}(A + B).$$

Since $\frac{1}{2}(A + B) = \frac{1}{2}(180° - C)$, the value of $\frac{1}{2}(A + B)$ is known, so that this equation enables us to find the value of $\frac{1}{2}(A - B)$. We then have

$$\tfrac{1}{2}(A + B) + \tfrac{1}{2}(A - B) = A$$

and

$$\tfrac{1}{2}(A + B) - \tfrac{1}{2}(A - B) = B.$$

After A and B are known, the side c may be found by the Law of Sines, which gives its value in two ways, as follows:

$$c = \frac{a \sin C}{\sin A}, \quad \text{or} \quad c = \frac{b \sin C}{\sin B}.$$

Solution II. The third side c may be found directly from the equation (Sect. XXXV, p. 66)

$$c = \sqrt{a^2 + b^2 - 2ab \cos C};$$

and then, by the Law of Sines, the following equations for computing the values of the angles A and B are obtained:

$$\sin A = a \times \frac{\sin C}{c}. \quad \sin B = b \times \frac{\sin C}{c}.$$

SOLUTION III. If, in the triangle ABC (Fig. 62), BD is drawn perpendicular to the side AC, then

$$\tan A = \frac{BD}{AD} = \frac{BD}{AC - DC}.$$

Now $BD = a \sin C$

and $DC = a \cos C.$

$$\therefore \tan A = \frac{a \sin C}{b - a \cos C}.$$

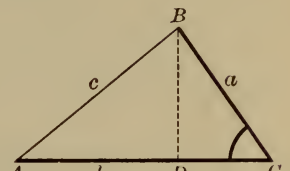


FIG. 62

By merely changing the letters,

$$\tan B = \frac{b \sin C}{a - b \cos C}.$$

It is not necessary, however, to use both formulas. When one angle, as A, has been found, the other, B, may be found from the relation $A + B + C = 180°$.

When the angles are known, the third side is found by the Law of Sines, as in Solution I.

NOTE. When all three unknown parts are required, Solution I is the most convenient in practice. When only the third side, c, is desired, Solution II may be used to advantage, provided the values of a^2 and b^2 can be obtained readily without the aid of logarithms. But Solutions II and III are not adapted to logarithmic work.

EXAMPLE 1. Given $a = 748$, $b = 375$, $C = 63° 35' 30''$; find A, B, and c.

$a + b = 1123$		
$a - b = 373$	$\log (a - b) = 2.57171$	$\log b = 2.57403$
$A + B = 116° 24' 30''$	$\operatorname{colog} (a + b) = 6.94962$	$\log \sin C = 9.95214$
$\frac{1}{2}(A + B) = 58° 12' 15''$	$\log \tan \frac{1}{2}(A + B) = 0.20766$	$\operatorname{colog} \sin B = 0.30073$
$\frac{1}{2}(A - B) = 28° 10' 54''$	$\log \tan \frac{1}{2}(A - B) = 9.72899$	$\log c = 2.82690$
$A = 86° 23' 9'$	$\frac{1}{2}(A - B) = 28° 10' 54''$	$c = 671.27$
$B = 30° 1' 21''$		

NOTE. In the above example we use the angle B in finding the side c rather than the angle A, because A is near 90°, and therefore the use of its sine should be avoided. See Note, p. 23.

EXAMPLE 2. Given $a = 4$, $c = 6$, $B = 60°$; find the third side b.

Here Solution II may be used to advantage. We have

$$b = \sqrt{a^2 + c^2 - 2\,ac\cos B} = \sqrt{16 + 36 - 24} = \sqrt{28};$$

$$\log 28 = 1.44716, \qquad \log \sqrt{28} = 0.72358, \qquad \sqrt{28} = 5.2915;$$

that is, $b = 5.2915$.

EXERCISE XIX

1. Given $a = 77.99$, $b = 83.39$, $C = 72° 15'$;
   find $A = 51° 15'$, $B = 56° 30'$, $c = 95.24$.

2. Given $b = 872.5$, $c = 632.7$, $A = 80°$;
   find $B = 60° 45' 2''$, $C = 39° 14' 58''$, $a = 984.83$.

3. Given $a = 17$, $b = 12$, $C = 59° 17'$;
   find $A = 77° 12' 53''$, $B = 43° 30' 7''$, $c = 14.987$.

4. Given $b = \sqrt{5}$, $c = \sqrt{3}$, $A = 35° 53'$;
   find $B = 93° 28' 36''$, $C = 50° 38' 24''$, $a = 1.3131$.

5. Given $a = 0.917$, $b = 0.312$, $C = 33° 7' 9''$;
   find $A = 132° 18' 27''$, $B = 14° 34' 24''$, $c = 0.6775$.

6. Given $a = 13.715$, $c = 11.214$, $B = 15° 22' 36''$;
   find $A = 118° 55' 49''$, $C = 45° 41' 35''$, $b = 4.1554$.

7. Given $b = 3000.9$, $c = 1587.2$, $A = 86° 4' 4''$;
   find $B = 65° 13' 51''$, $C = 28° 42' 5''$, $a = 3297.2$.

8. Given $a = 4527$, $b = 3465$, $C = 66° 6' 27''$;
   find $A = 68° 29' 15''$, $B = 45° 24' 18''$, $c = 4449$.

9. Given $a = 55.14$, $b = 33.09$, $C = 30° 24'$;
   find $A = 117° 24' 32''$, $B = 32° 11' 28''$, $c = 31.431$.

10. Given $a = 47.99$, $b = 33.14$, $C = 175° 19' 10''$;
    find $A = 2° 46' 8''$, $B = 1° 54' 42''$, $c = 81.066$.

11. If two sides of a triangle are each equal to 6, and the included angle is 60°, find the third side.

12. If two sides of a triangle are each equal to 6, and the included angle is 120°, find the third side.

13. Apply Solution I to the case in which a is equal to b; that is, the case in which the triangle is isosceles.

14. If two sides of a triangle are 10 and 11, and the included angle is 50°, find the third side.

15. If two sides of a triangle are 43.301 and 25, and the included angle is 30°, find the third side.

16. In order to find the distance between two objects, A and B, separated by a swamp, a station C was chosen, and the distances $CA = 3825$ yards, $CB = 3475.6$ yards, together with the angle $ACB = 62° 31'$, were measured. Find the distance from A to B.

17. Two inaccessible objects, A and B, are each viewed from two stations, C and D, on the same side of AB and 562 yards apart. The angle ACB is 62° 12', BCD 41° 8', ADB 60° 49', and ADC 34° 51'; required the distance AB.

18. Two trains start at the same time from the same station and move along straight tracks that form an angle of 30°, one train at the rate of 30 miles an hour, the other at the rate of 40 miles an hour. How far apart are the trains at the end of half an hour?

19. In a parallelogram given the two diagonals 5 and 6, and the angle that they form 49° 18'; find the sides.

20. In a triangle one angle is 139° 54', and the sides forming the angle have the ratio 5 : 9. Find the other two angles.

21. In order to find the distance between two objects, A and B, separated by a pond, a station C was chosen, and the distances $CA = 426$ yards, $CB = 322.4$ yards, together with the angle $ACB = 68° 42'$, were measured. Find the distance from A to B.

SECTION XLI

Case IV

Given the three sides a, b, c; *find the angles* A, B, C.

The angles may be found directly from the formulas established in Sect. XXXV, p. 66. Thus, from the formula

$$a^2 = b^2 + c^2 - 2bc\cos A,$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

From this equation formulas adapted to logarithmic work are deduced as follows:

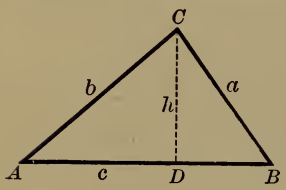


FIG. 63

For the sake of brevity, let

$$a + b + c = 2s;$$

then $$b + c - a = 2(s - a),$$

$$a - b + c = 2(s - b),$$

and $$a + b - c = 2(s - c).$$

Then the value of $1 - \cos A$ is

$$1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{2bc - b^2 - c^2 + a^2}{2bc} = \frac{a^2 - (b - c)^2}{2bc}$$

$$= \frac{(a + b - c)(a - b + c)}{2bc}$$

$$= \frac{2(s - b)(s - c)}{bc},$$

and the value of $1 + \cos A$ is

$$1 + \frac{b^2 + c^2 - a^2}{2bc} = \frac{2bc + b^2 + c^2 - a^2}{2bc} = \frac{(b + c)^2 - a^2}{2bc}$$

$$= \frac{(b + c + a)(b + c - a)}{2bc} = \frac{2s(s - a)}{bc}.$$

But from Formulas [16] and [17], p. 58, it follows that

$$1 - \cos A = 2\sin^2 \tfrac{1}{2} A, \text{ and } 1 + \cos A = 2\cos^2 \tfrac{1}{2} A.$$

$$\therefore 2\sin^2 \tfrac{1}{2} A = \frac{2(s-b)(s-c)}{bc}, \text{ and } 2\cos^2 \tfrac{1}{2} A = \frac{2s(s-a)}{bc},$$

whence

$$\sin \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{bc}}, \qquad [28]$$

$$\cos \tfrac{1}{2} A = \sqrt{\frac{s(s-a)}{bc}}, \qquad [29]$$

and by [2],

$$\tan \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}. \qquad [30]$$

By merely changing the letters,

$$\sin \tfrac{1}{2} B = \sqrt{\frac{(s-a)(s-c)}{ac}}, \quad \sin \tfrac{1}{2} C = \sqrt{\frac{(s-a)(s-b)}{ab}}.$$

$$\cos \tfrac{1}{2} B = \sqrt{\frac{s(s-b)}{ac}}, \quad \cos \tfrac{1}{2} C = \sqrt{\frac{s(s-c)}{ab}}.$$

$$\tan \tfrac{1}{2} B = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}, \quad \tan \tfrac{1}{2} C = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

There is then a choice of three different formulas for finding the value of each angle. If half the angle is very near 0°, the formula for the cosine will not give a very accurate result, because the cosines of angles near 0° differ little in value; and the same holds true of the formula for the sine when half the angle is very near 90°. Hence, in the first case the formula for the sine, in the second that for the cosine, should be used.

But, in general, the formulas for the tangent are to be preferred.

It is not necessary to compute by the formulas more than two angles; for the third may then be found from the equation

$$A + B + C = 180°.$$

There is this advantage, however, in computing all three angles by the formulas, that we may then use the sum of the angles as a test of the accuracy of the results.

In case it is desired to compute all the angles, the formulas for the tangent may be put in a more convenient form.

The value of $\tan \frac{1}{2} A$ may be written

$$\sqrt{\frac{(s-a)(s-b)(s-c)}{s(s-a)^2}}$$

or

$$\frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

Hence, if we put

$$\sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = r, \qquad [31]$$

we have

$$\tan \tfrac{1}{2} A = \frac{r}{s-a}. \qquad [32]$$

Likewise, $\tan \frac{1}{2} B = \frac{r}{s-b}$, $\tan \frac{1}{2} C = \frac{r}{s-c}$.

EXAMPLE 1. Given $a = 3.41$, $b = 2.60$, $c = 1.58$; find the angles.

Using Formula [30] and the corresponding formula for $\tan \frac{1}{2} B$, we may arrange the work as follows:

$a = 3.41$	$\operatorname{colog} s = 9.42079$	$\operatorname{colog} s = 9.42079 - 10$
$b = 2.60$	$\operatorname{colog}(s-a) = 0.41454$	$\log(s-a) = 9.58546 - 10$
$c = 1.58$	$\log(s-b) = 0.07737$	$\operatorname{colog}(s-b) = 9.92263 - 10$
$2s = 7.59$	$\log(s-c) = 0.34537$	$\log(s-c) = 0.34537$
$s = 3.795$	$2 \overline{)0.25807}$	$2 \overline{)19.27425 - 20}$
$s-a = 0.385$	$\log \tan \frac{1}{2} A = 0.12903$	$\log \tan \frac{1}{2} B = 9.63713 - 10$
$s-b = 1.195$	$\frac{1}{2} A = 53° 23' 20''$	$\frac{1}{2} B = 23° 26' 37''$
$s-c = 2.215$	$A = 106° 46' 40''$	$B = 46° 53' 14''$

$\therefore A + B = 153° 39' 54''$, and $C = 26° 20' 6''$.

EXAMPLE 2. Solve Example 1 by finding all three angles by the use of Formulas [31] and [32].

Here the work may be compactly arranged as follows, if we find $\log \tan \frac{1}{2}A$, etc., by *subtracting* $\log (s - a)$, etc., from $\log r$ instead of adding the cologarithm:

$a = 3.41$	$\log (s - a) = 9.58546$	$\log \tan \frac{1}{2}A = 10.12903$
$b = 2.60$	$\log (s - b) = 0.07737$	$\log \tan \frac{1}{2}B = 9.63713$
$c = 1.58$	$\log (s - c) = 0.34537$	$\log \tan \frac{1}{2}C = 9.36912$
$2s = 7.59$	$\operatorname{colog} s = 9.42079$	$\frac{1}{2}A = 53° 23' 20''$
$s = 3.795$	$\log r^2 = 9.42899$	$\frac{1}{2}B = 23° 26' 37''$
$s - a = 0.385$	$\log r = 9.71450$	$\frac{1}{2}C = 13° 10' 3''$
$s - b = 1.195$		$A = 106° 46' 40''$
$s - c = 2.215$		$B = 46° 53' 14''$
$2s = 7.590$ (check).		$C = 26° 20' 6''$
		Check, $A + B + C = 180° 0' 0''$

NOTE. Even if no mistakes are made in the work, the sum of the three angles found as above may differ very slightly from 180° in consequence of the fact that logarithmic computation is at best only a method of close approximation. When a difference of this kind exists, it should be divided among the angles according to the probable amount of error for each angle.

EXERCISE XX

Solve the following triangles, taking the three sides as the given parts:

	a	b	c	A	B	C
1	51	65	20	38° 52′ 48″	126° 52′ 12″	14° 15′
2	78	101	29	32° 10′ 55″	136° 23′ 50″	11° 25′ 15″
3	111	145	40	27° 20′ 32″	143° 7′ 48″	9° 31′ 40″
4	21	26	31	42° 6′ 13″	56° 6′ 36″	81° 47′ 11″
5	19	34	49	16° 25′ 36″	30° 24′	133° 10′ 24″
6	43	50	57	46° 49′ 35″	57° 59′ 44″	75° 10′ 41″
7	37	58	79	26° 0′ 29″	43° 25′ 20″	110° 34′ 11″
8	73	82	91	49° 34′ 58″	58° 46′ 58″	71° 38′ 4″
9	14.493	55.4363	66.9129	8° 20′ 1″	33° 40′ 5″	137° 59′ 54″
10	$\sqrt{5}$	$\sqrt{6}$	$\sqrt{7}$	51° 53′ 12″	59° 31′ 48″	68° 35′

11. Given $a = 6$, $b = 8$, $c = 10$; find the angles.

12. Given $a = 6$, $b = 6$, $c = 10$; find the angles.

13. Given $a = 6$, $b = 6$, $c = 6$; find the angles.

14. Given $a = 6$, $b = 9$, $c = 12$; find the angles.

15. Given $a = 2$, $b = \sqrt{6}$, $c = \sqrt{3} - 1$; find the angles.

16. Given $a = 2$, $b = \sqrt{6}$, $c = \sqrt{3} + 1$; find the angles.

17. The distances between three cities, A, B, and C, are as follows: $AB = 165$ miles, $AC = 72$ miles, and $BC = 185$ miles. B is due east from A. In what direction is C from A? What two answers are admissible?

18. Under what visual angle is an object 7 feet long seen by an observer whose eye is 5 feet from one end of the object and 8 feet from the other end?

19. When Formula [28] is used for finding the value of an angle, why does the ambiguity that occurs in Case II not exist?

20. If the sides of a triangle are 3, 4, and 6, find the sine of the largest angle.

21. Of three towns, A, B, and C, A is 200 miles from B and 184 miles from C, B is 150 miles due north from C. How far is A north of C?

22. The sides of a triangle are 78.9, 65.4, 97.3, respectively. Find the largest angle.

23. The sides of a triangle are 487.25, 512.33, 544.37, respectively. Find the smallest angle.

24. Find the angles of a triangle whose sides are $\frac{\sqrt{3}+1}{2\sqrt{2}}$, $\frac{\sqrt{3}-1}{2\sqrt{2}}$, $\frac{\sqrt{3}}{2}$, respectively.

25. The sides of a triangle are 14.6 inches, 16.7 inches, and 18.8 inches, respectively. Find the length of the perpendicular from the vertex of the largest angle upon the opposite side.

SECTION XLII

AREA OF A TRIANGLE

Case I

When two sides and the included angle are given.

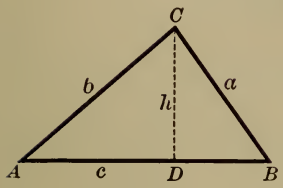


Fig. 64

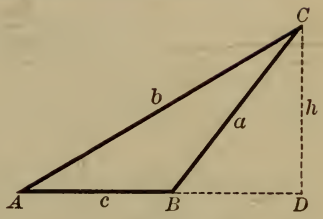


Fig. 65

In the triangle ABC (Fig. 64 or 65),

$$F = \tfrac{1}{2} c \times CD.$$

Now, $$CD = a \sin B.$$

Therefore, $$\mathbf{F = \tfrac{1}{2} ac \sin B.} \qquad [33]$$

Also, $F = \tfrac{1}{2} ab \sin C$, and $F = \tfrac{1}{2} bc \sin A$.

Case II

When a side and the two adjacent angles are given.

$$\sin A : \sin C = a : c. \quad \text{(Sect. XXXIV, p. 65.)}$$

Therefore, $$c = \frac{a \sin C}{\sin A}.$$

Putting this value of c in Formula [33],

$$F = \frac{a^2 \sin B \sin C}{2 \sin A}.$$

But $\sin (B + C) = \sin (180° - A) = \sin A$. (Sect. XXV, p. 48.)

Hence, $$\mathbf{F = \frac{a^2 \sin B \sin C}{2 \sin (B + C)}}. \qquad [34]$$

Case III

When the three sides of a triangle are given.

By Formula [12], p. 58,

$$\sin B = 2 \sin \tfrac{1}{2} B \times \cos \tfrac{1}{2} B.$$

Now, by Formula [28], p. 81,

$$\sin \tfrac{1}{2} B = \sqrt{\frac{(s-a)(s-c)}{ac}},$$

and by Formula [29], p. 81,

$$\cos \tfrac{1}{2} B = \sqrt{\frac{s(s-b)}{ac}}.$$

By substituting these values of $\sin \frac{1}{2} B$ and $\cos \frac{1}{2} B$ in the above equation, we have

$$\sin B = \frac{2}{ac} \sqrt{s(s-a)(s-b)(s-c)}.$$

By putting this value of $\sin B$ in [33], we have

$$\mathbf{F = \sqrt{s(s-a)(s-b)(s-c)}}. \qquad [35]$$

Case IV

When the three sides and the radius of the circumscribed circle or the radius of the inscribed circle are given.

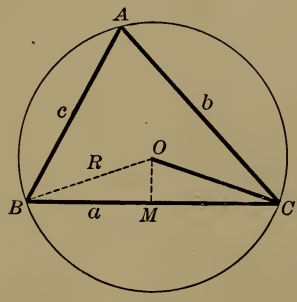


Fig. 66

If R denotes the radius of the circumscribed circle, we have, from Sect. XXXIV, p. 65,

$$\sin B = \frac{b}{2R}.$$

By putting this value of $\sin B$ in [33], we have

$$\mathbf{F = \frac{abc}{4R}}. \qquad [36]$$

If r denotes the radius of the inscribed circle, divide the triangle into three triangles by lines from the centre of this circle to the vertices; then the altitude of each of the three triangles is equal to r. Therefore,

$$\mathbf{F = \tfrac{1}{2} r\,(a + b + c) = rs.} \qquad [37]$$

By putting in this formula the value of F given in [35],

$$r = \sqrt{\frac{(s-a)\,(s-b)\,(s-c)}{s}};$$

whence r, in [31], Sect. XLI, p. 82, is equal to the radius of the inscribed circle.

EXERCISE XXI

Find the area:

1. Given $a = 4474.5$, $b = 2164.5$, $C = 116° 30' 20''$.
2. Given $b = 21.66$, $c = 36.94$, $A = 66° 4' 19''$.
3. Given $a = 510$, $c = 173$, $B = 162° 30' 28''$.
4. Given $a = 408$, $b = 41$, $c = 401$.
5. Given $a = 40$, $b = 13$, $c = 37$.
6. Given $a = 624$, $b = 205$, $c = 445$.
7. Given $b = 149$, $A = 70° 42' 30''$, $B = 39° 18' 28''$.
8. Given $a = 215.9$, $c = 307.7$, $A = 25° 9' 31''$.
9. Given $b = 8$, $c = 5$, $A = 60°$.
10. Given $a = 7$, $c = 3$, $A = 60°$.

11. Given $a = 60$, $B = 40° 35' 12''$, area $= 12$; find the radius of the inscribed circle.

12. Obtain a formula for the area of a parallelogram in terms of two adjacent sides and the included angle.

13. Obtain a formula for the area of an isosceles trapezoid in terms of the two parallel sides and an acute angle.

14. Two sides and included angle of a triangle are 2416, 1712, and 30°; and two sides and included angle of another triangle are 1948, 2848, and 150°. Find the sum of their areas.

15. The base of an isosceles triangle is 20, and its area is $100 \div \sqrt{3}$; find its angles.

16. Show that the area of a quadrilateral is equal to one-half the product of its diagonals into the sine of their included angle.

EXERCISE XXII

1. From a ship sailing down the English Channel the Eddystone was observed to bear N. 33° 45′ W., and after the ship had sailed 18 miles S. 67° 30′ W. it bore N. 11° 15′ E. Find its distance from each position of the ship.

2. Two objects, A and B, were observed from a ship to be at the same instant in a line bearing N. 15° E. The ship then sailed northwest 5 miles, when it was found that A bore due east and B bore northeast. Find the distance from A to B.

3. A castle and a monument stand on the same horizontal plane. The angles of depression of the top and the bottom of the monument viewed from the top of the castle are 40° and 80°; the height of the castle is 140 feet. Find the height of the monument.

4. If the sun's altitude is 60°, what angle must a stick make with the horizon in order that its shadow in a horizontal plane may be the longest possible?

5. If the sun's altitude is 30°, find the length of the longest shadow cast on a horizontal plane by a stick 10 feet in length.

6. In a circle with the radius 3 find the area of the part comprised between parallel chords whose lengths are 4 and 5. (Two solutions.)

CHAPTER V

MISCELLANEOUS EXAMPLES

Problems in Plane Trigonometry

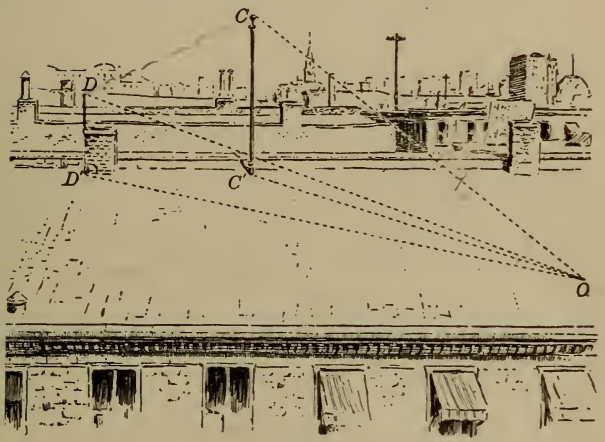


Fig. 67

If two objects are not in the same horizontal plane with each other or with the point of observation, we may suppose vertical lines to be passed through the two objects and to meet the horizontal plane of the point of observation in two points. The angular distance of these two points is the *bearing* of either of the objects from the other. Thus, the angle $C'OD'$ (Fig. 67) is the bearing of C from D.

Note. "Problems in Plane Trigonometry" are selected from those published by Mr. Charles W. Seaver, Cambridge, Mass. The full set can be obtained from him in pamphlet form.

EXERCISE XXIII

RIGHT TRIANGLES

1. The angle of elevation of a tower is 48° 19′ 14″, and the distance of the base from the point of observation is 95 feet. Find the height of the tower and the distance of its top from the point of observation.

2. From a mountain 1000 feet high, the angle of depression of a ship is 77° 35′ 11″. Find the distance of the ship from the summit of the mountain.

3. A flagstaff 90 feet high, on a horizontal plane, casts a shadow of 117 feet. Find the altitude of the sun.

4. When the moon is setting at any place, the angle at the moon subtended by the earth's radius passing through that place is 57′ 3″. If the earth's radius is 3956.2 miles, what is the moon's distance from the earth's centre?

5. The angle at the earth's centre subtended by the sun's radius is 16′ 2″ and the sun's distance is 92,400,000 miles. Find the sun's diameter in miles.

6. The latitude of Cambridge, Mass., is 42° 22′ 49″. What is the length of the radius of that parallel of latitude?

7. At what latitude is the circumference of the parallel of latitude half of that of the equator?

8. In a circle with a radius of 6.7 is inscribed a regular polygon of thirteen sides. Find the length of one of its sides.

9. A regular heptagon, one side of which is 5.73, is inscribed in a circle. Find the radius of the circle.

10. A tower 93.97 feet high is situated on the bank of a river. The angle of depression of an object on the opposite bank is 25° 12′ 54″. Find the breadth of the river.

11. From a tower 58 feet high the angles of depression of two objects situated in the same horizontal line with the base of the tower, and on the same side, are 30° 13′ 18″ and 45° 46′ 14″. Find the distance between these two objects.

12. Standing directly in front of one corner of a flat-roofed house, which is 150 feet in length, I observe that the horizontal angle which the length subtends has for its cosine $\sqrt{\frac{1}{3}}$, and that the vertical angle subtended by its height has for its sine $\frac{3}{\sqrt{34}}$. What is the height of the house?

13. A regular pyramid, with a square base, has a lateral edge 150 feet long, and a side of its base is 200 feet. Find the inclination of the face of the pyramid to the base.

14. From one edge of a ditch 36 feet wide, the angle of elevation of a wall on the opposite edge is 62° 39′ 10″. Find the length of a ladder that will just reach from the point of observation to the top of the wall.

15. The top of a flagstaff has been partly broken off and touches the ground at a distance of 15 feet from the foot of the staff. If the length of the broken part is 39 feet, find the length of the whole staff.

16. From a balloon, which is directly above one town, is observed the angle of depression of another town, 10° 14′ 9″. The towns being 8 miles apart, find the height of the balloon.

17. From the top of a mountain 3 miles high the angle of depression of the most distant object which is visible on the earth's surface is found to be 2° 13′ 50″. Find the diameter of the earth.

18. A ladder 40 feet long reaches a window 33 feet high, on one side of a street. Being turned over upon its foot, it reaches another window 21 feet high, on the opposite side of the street. Find the width of the street.

19. The height of a house subtends a right angle at a window on the other side of the street; and the angle of elevation of the top of the house, from the same point, is 60°. The street is 30 feet wide. How high is the house?

20. A lighthouse 54 feet high is situated on a rock. The angle of elevation of the top of the lighthouse, as observed from a ship, is 4° 52′, and the angle of elevation of the top of the rock is 4° 2′. Find the height of the rock and its distance from the ship.

21. A man in a balloon observes the angle of depression of an object on the ground, bearing south, to be 35° 30′; the balloon drifts $2\frac{1}{2}$ miles east at the same height, when the angle of depression of the same object is 23° 14′. Find the height of the balloon.

22. A man standing south of a tower, on the same horizontal plane, observes its angle of elevation to be 54° 16′; he goes east 100 yards, and then finds its angle of elevation is 50° 8′. Find the height of the tower.

23. The angle of elevation of a tower at a place A, south of it, is 30°; and at a place B, west of A, and at a distance of a from it, the angle of elevation is 18°. Show that the height of the tower is $\frac{a}{\sqrt{2+2\sqrt{5}}}$; the tangent of 18° being $\frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$.

24. A pole is fixed on the top of a mound, and the angles of elevation of the top and the bottom of the pole are 60° and 30°, respectively. Prove that the length of the pole is twice the height of the mound.

25. At a distance a from the foot of a tower, the angle of elevation A of the top of the tower is the complement of the angle of elevation of a flagstaff on top of it. Show that the length of the staff is $2\,a \cot 2\,A$.

26. *A line of true level* is a line every point of which is equally distant from the centre of the earth. A line drawn tangent to a line of true level at any point is a line of *apparent* level. If at any point both these lines are drawn, and extended one mile, find the distance they are then apart.

27. In Problem 1, page 90, determine the effect upon the computed height of the tower of an error in either the angle of elevation or the measured distance.

OBLIQUE TRIANGLES

28. To determine the height of an inaccessible object situated on a horizontal plane, by observing its angles of elevation at two points in the same line with its base, and measuring the distance between these two points.

29. The angle of elevation of an inaccessible tower situated on a horizontal plane is 63° 26′; at a point 500 feet farther from the base of the tower the angle of elevation of its top is 32° 14′. Find the height of the tower.

30. A tower is situated on the bank of a river. From the opposite bank the angle of elevation of the tower is 60° 13′, and from a point 40 feet more distant the angle of elevation is 50° 19′. Find the breadth of the river.

31. A ship sailing north sees two lighthouses 8 miles apart, in a line due west; after an hour's sailing, one lighthouse bears S.W., and the other S.S.W. Find the ship's rate.

32. To determine the height of an accessible object situated on an inclined plane.

33. At the distance of 40 feet from the foot of a tower on an inclined plane, the tower subtends an angle of 41° 19′; at a point 60 feet farther away, the angle subtended by the tower is 23° 45′. Find the height of the tower.

34. A tower makes an angle of 113° 12′ with the inclined plane on which it stands ; and at a distance of 89 feet from its base, measured down the plane, the angle subtended by the tower is 23° 27′. Find the height of the tower.

35. From the top of a house 42 feet high the angle of elevation of the top of a pole is 14° 13′; at the bottom of the house it is 23° 19′. Find the height of the pole.

36. The sides of a triangle are 17, 21, 28. Prove that the length of a line bisecting the greatest side and drawn from the opposite angle is 13.

37. A privateer, 10 miles S.W. of a harbor, sees a ship sail from it in a direction S. 80° E., at a rate of 9 miles an hour. In what direction, and at what rate, must the privateer sail in order to come up with the ship in $1\frac{1}{2}$ hours ?

38. A person goes 70 yards up a slope of 1 in $3\frac{1}{2}$ from the edge of a river and observes the angle of depression of an object on the opposite bank to be $2\frac{1}{4}°$. Find the breadth of the river.

39. The length of a lake subtends, at a certain point, an angle of 46° 24′, and the distances from this point to the two extremities of the lake are 346 and 290 feet. Find the length of the lake.

40. Two ships are a mile apart. The angular distance of the first ship from a fort on shore, as observed from the second ship, is 35° 14′ 10″; the angular distance of the second ship from the fort, observed from the first ship, is 42° 11′ 53″. Find the distance in feet from each ship to the fort.

41. Along the bank of a river is drawn a base line of 500 feet. The angular distance of one end of this line from an object on the opposite side of the river, as observed from the other end of the line, is 53° ; that of the second extremity

from the same object, observed at the first, is 79° 12′. Find the breadth of the river.

42. A vertical tower stands on a declivity inclined 15° to the horizon. A man ascends the declivity 80 feet from the base of the tower, and finds the angle then subtended by the tower to be 30°. Find the height of the tower.

43. The angle subtended by a tower on an inclined plane is, at a certain point, 42° 17′; 325 feet farther down it is 21° 47′. The inclination of the plane is 8° 53′. Find the height of the tower.

44. A cape bears north by east, as seen from a ship. The ship sails northwest 30 miles, and then the cape bears east. How far is it from the second point of observation?

45. Two observers, stationed on *opposite* sides of a cloud, observe its angles of elevation to be 44° 56′ and 36° 4′. Their distance from each other is 700 feet. What is the height of the cloud?

46. From a point B at the foot of a mountain, the angle of elevation of the top A is 60°. After ascending the mountain one mile, at an inclination of 30° to the horizon, and reaching a point C, the angle ACB is found to be 135°. Find the height of the mountain in feet.

47. From a ship two rocks are seen in the same right line with the ship, bearing N. 15° E. After the ship has sailed northwest 5 miles, the first rock bears east, and the second northeast. Find the distance between the rocks.

48. From a window on a level with the bottom of a steeple the angle of elevation of the steeple is 40°, and from a second window 18 feet higher the angle of elevation is 37° 30′. Find the height of the steeple.

49. To determine the distance between two inaccessible objects by observing angles at the extremities of a line of known length.

50. Wishing to determine the distance between a church A and a tower B, on the opposite side of a river, I measure a line CD along the river (C being nearly opposite A), and observe the angles ACB, 58° 20′; ACD, 95° 20′; ADB, 53° 30′; BDC, 98° 45′. CD is 600 feet. What is the distance required?

51. Wishing to find the height of a summit A, I measure a horizontal base line CD, 440 yards. At C, the angle of elevation of A is 37° 18′, and the horizontal angle between D and the summit is 76° 18′; at D, the horizontal angle between C and the summit is 67° 14′. Find the height.

52. A balloon is observed from two stations 3000 feet apart. At the first station the horizontal angle of the balloon and the other station is 75° 25′, and the angle of elevation of the balloon is 18°. The horizontal angle of the first station and the balloon, measured at the second station, is 64° 30′. Find the height of the balloon.

53. Two forces, one of 410 pounds, and the other of 320 pounds, make an angle of 51° 37′. Find the intensity and the direction of their resultant.

54. An unknown force, combined with one of 128 pounds, produces a resultant of 200 pounds, and this resultant makes an angle of 18° 24′ with the known force. Find the intensity and direction of the unknown force.

55. At two stations, the height of a kite subtends the same angle A. The angle which the line joining one station and the kite subtends at the other station is B; and the distance between the two stations is a. Show that the height of the kite is $\frac{1}{2}\, a \sin A \sec B$.

56. Two towers on a horizontal plane are 120 feet apart. A person standing successively at their bases observes that the angle of elevation of one is double that of the other; but, when he is half-way between them, the angles of elevation are complementary. Prove that the heights of the towers are 90 and 40 feet.

57. To find the distance of an inaccessible point C from either of two points A and B, having no instruments to measure angles. Prolong CA to a, and CB to b, and join AB, Ab, and Ba. Measure AB, 500; aA, 100; aB, 560; bB, 100; and Ab, 550. Compute the distances AC and BC.

58. Two inaccessible points A and B are visible from D, but no other point can be found whence both are visible. Take some point C, whence A and D can be seen, and measure CD, 200 feet; ADC, 89°; ACD, 50° 30′. Then take some point E, whence D and B are visible, and measure DE, 200 feet; BDE, 54° 30′; BED, 88° 30′. At D measure ADB, 72° 30′. Compute the distance AB.

59. To compute the horizontal distance between two inaccessible points A and B, when no point can be found whence both can be seen. Take two points C and D, distant 200 yards, so that A can be seen from C, and B from D. From C measure CF, 200 yards to F, whence A can be seen; and from D measure DE, 200 yards to E, whence B can be seen. Measure AFC, 83°; ACD, 53° 30′; ACF, 54° 31′; BDE, 54° 30′; BDC, 156° 25′; DEB, 88° 30′.

60. A column in the north temperate zone is east-southeast of an observer, and at noon the extremity of its shadow is northeast of him. The shadow is 80 feet in length, and the elevation of the column, at the observer's station, is 45°. Find the height of the column.

61. From the top of a hill the angles of depression of two objects situated in the horizontal plane of the base of the hill are 45° and 30°; and the horizontal angle between the two objects is 30°. Show that the height of the hill is equal to the distance between the objects.

62. Wishing to know the breadth of a river from A to B, I take AC, 100 yards in the prolongation of BA, and then take CD, 200 yards at right angles to AC. The angle BDA is 37° 18′ 30″. Find AB.

63. The sum of the sides of a triangle is 100. The angle at A is double that at B, and the angle at B is double that at C. Determine the sides.

64. If $\sin^2 A + 5 \cos^2 A = 3$, find A.

65. If $\sin^2 A = m \cos A - n$, find $\cos A$.

66. Given $\sin A = m \sin B$, and $\tan A = n \tan B$; find $\sin A$ and $\cos B$.

67. If $\tan^2 A + 4 \sin^2 A = 6$, find A.

68. If $\sin A = \sin 2A$, find A.

69. If $\tan 2A = 3 \tan A$, find A.

70. Prove that $\tan 50° + \cot 50° = 2 \sec 10°$.

71. Given a regular polygon of n sides, and calling one of them a, find expressions for the radii of the inscribed and the circumscribed circles in terms of n and a.

If P, H, D are the sides of a regular inscribed pentagon, hexagon, and decagon, prove $P^2 = H^2 + D^2$.

AREAS

72. Obtain the formula for the area of a triangle, given two sides b, c, and the included angle A.

73. Obtain the formula for the area of a triangle, given two angles A and B, and included side c.

74. Obtain the formula for the area of a triangle, given the three sides.

75. If a is the side of an equilateral triangle, show that its area is $\frac{a^2\sqrt{3}}{4}$.

76. Two consecutive sides of a rectangle are 52.25 chains and 38.24 chains. Find the area.

77. Two sides of a parallelogram are 59.8 chains and 37.05 chains, and the included angle is 72° 10′. Find the area.

78. Two sides of a parallelogram are 15.36 chains and 11.46 chains, and the included angle is 47° 30′. Find the area.

79. Two sides of a triangle are 12.38 chains and 6.78 chains, and the included angle is 46° 24′. Find the area.

80. Two sides of a triangle are 18.37 chains and 13.44 chains, and they form a right angle. Find the area.

81. Two angles of a triangle are 76° 54′ and 57° 33′ 12″, and the included side is 9 chains. Find the area.

82. Two sides of a triangle are 19.74 chains and 17.34 chains. The first bears N. 82° 30′ W.; the second S. 24° 15′ E. Find the area.

83. The three sides of a triangle are 49 chains, 50.25 chains, and 25.69 chains. Find the area.

84. The three sides of a triangle are 10.64 chains, 12.28 chains, and 9 chains. Find the area.

85. The sides of a triangular field, of which the area is 14 acres, are in the ratio of 3, 5, 7. Find the sides.

86. In the quadrilateral $ABCD$ we have AB, 17.22 chains; AD, 7.45 chains; CD, 14.10 chains; BC, 5.25 chains; and the diagonal AC, 15.04 chains. Find the area.

L. of C.

87. The diagonals of a quadrilateral are a and b, and they intersect at an angle D. Show that the area of the quadrilateral is $\frac{1}{2}ab \sin D$.

88. The diagonals of a quadrilateral are 34 and 56, intersecting at an angle of 67°. Find the area.

89. The diagonals of a quadrilateral are 75 and 49, intersecting at an angle of 42°. Find the area.

90. Show that the area of a regular polygon of n sides, of which one is a, is $\frac{na^2}{4} \cot \frac{180°}{n}$.

91. One side of a regular pentagon is 25. Find the area.

92. One side of a regular hexagon is 32. Find the area.

93. One side of a regular decagon is 46. Find the area.

94. Find the area of a circle whose circumference is 74 feet.

95. Find the area of a circle whose radius is 125 feet.

96. In a circle with a diameter of 125 feet find the area of a sector with an arc of 22°.

97. In a circle with a radius of 44 feet find the area of a sector with an arc of 25°.

98. In a circle with a diameter of 50 feet find the area of a segment with an arc of 280°.

99. Find the area of a segment (less than a semicircle) of which the chord is 20, and the distance of the chord from the middle point of the smaller arc is 2.

100. If r is the radius of a circle, the area of a regular circumscribed polygon of n sides is $nr^2 \tan \frac{180°}{n}$.

The area of a regular inscribed polygon is $\frac{n}{2} r^2 \sin \frac{360°}{n}$.

101. If a is a side of a regular polygon of n sides, the area of the inscribed circle is $\frac{\pi a^2}{4} \cot^2 \frac{180°}{n}$.

The area of the circumscribed circle is $\frac{\pi a^2}{4} \csc^2 \frac{180°}{n}$.

102. The area of a regular polygon inscribed in a circle is to that of the circumscribed regular polygon of the same number of sides as 3 to 4. Find the number of sides.

103. The area of a regular polygon inscribed in a circle is the geometric mean between the areas of an inscribed and a circumscribed regular polygon of half the number of sides.

104. The area of a circumscribed regular polygon is the harmonic mean between the areas of an inscribed regular polygon of the same number of sides and of a circumscribed regular polygon of half that number.

105. The perimeter of a circumscribed regular triangle is double that of the inscribed regular triangle.

106. The square described about a circle is four-thirds the inscribed regular dodecagon.

107. Two sides of a triangle are 3 and 12, and the included angle is 30°. Find the hypotenuse of an isosceles right triangle of equal area.

PLANE SAILING

Plane Sailing is that branch of Navigation in which the surface of the earth is considered a plane. The problems which arise are therefore solved by the methods of Plane Trigonometry.

The *difference of latitude* of two places is the arc of a meridian comprehended between the parallels of latitude passing through those places.

The *departure* between two meridians is the arc of a parallel of latitude comprehended between those meridians. It diminishes as the distance from the equator increases.

When a ship sails in such a manner as to cross successive meridians at the same angle, it is said to sail on a *rhumb-line.* This angle is called the *course,* and the *distance* between two places is measured on a rhumb-line.

If we consider the distance, departure, and difference of latitude of two places to be straight lines, lying in one plane, they form a right triangle, called *the triangle of plane sailing.* If ABC is a plane triangle, right-angled at B, and BC represents the difference of latitude of B and C, ACB will be the course from C to A, CA the distance, and D the departure, measured from B, between the meridian of A and that of B.

108. Taking the earth's equatorial diameter to be 7925.6 miles, find the length in feet of the arc of one minute of a great circle.*

109. A ship sails from latitude 43° 45′ S., on a course N. by E. 2345 miles. Find the latitude reached, and the departure made.

110. A ship sails from latitude 1° 45′ N., on a course S.E. by E., and reaches latitude 2° 31′ S. Find the distance, and the departure.

111. A ship sails from latitude 13° 17′ S., on a course N.E. by E. $\frac{3}{4}$ E., until the departure is 207 miles. Find the distance, and the latitude reached.

112. A ship sails on a course between S. and E. 244 miles, leaving latitude 2° 52′ S., and reaching latitude 5° 8′ S. Find the course, and the departure.

* The length of the arc of one minute of a great circle of the earth is called a *geographical mile* or a *knot.* In the following problems, this is the distance meant by the term "mile," unless otherwise stated.

113. A ship sails from latitude 32° 18′ N., on a course between N. and W., a distance of 344 miles, and a departure of 103 miles. Find the course, and the latitude reached.

114. A ship sails on a course between S. and E., making a difference of latitude 136 miles, and a departure 203 miles. Find the distance, and the course.

115. A ship sails due north 15 *statute* miles an hour, for one day. What is the distance, in a straight line, from the point left to the point reached? (Take earth's radius, 3962.8 statute miles.)

PARALLEL AND MIDDLE LATITUDE SAILING

The *difference of longitude* of two places is the angle at the pole made by the meridians of these two places; or, it is the arc of the equator comprehended between these two meridians.

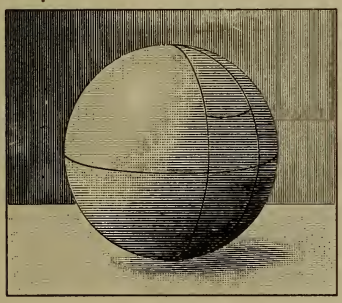


FIG. 68

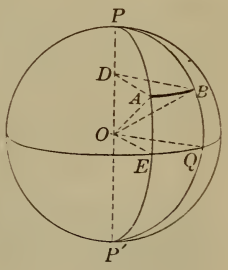


FIG. 69

In **Parallel Sailing** a vessel is supposed to sail due east or due west. The distance sailed is the departure made; and the difference of longitude is found as follows:

116. Given the departure between any two meridians at any latitude; find the difference of longitude of any point on one meridian from any point on the other.

SOLUTION. In rt. $\triangle ODA$, $\angle AOD = 90° - \text{lat.}$

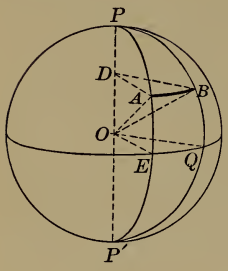


FIG. 70

Hence, $\dfrac{DA}{OA} = \sin(90° - \text{lat.}) = \cos \text{lat.}$

The $\triangle DAB$ and OEQ are similar.

Therefore, $\dfrac{DA}{OE} = \dfrac{AB}{EQ}$, or $\dfrac{DA}{OA} = \dfrac{AB}{EQ}$.

Hence, $\cos \text{lat.} = \dfrac{AB}{EQ}$.

Therefore, $EQ = \dfrac{AB}{\cos \text{lat.}} = AB \times \sec \text{lat.}$

That is, **Diff. long. = depart. × sec lat.**

117. A ship in latitude 42° 16′ N., longitude 72° 16′ W., sails due east a distance of 149 miles. What is the position of the point reached?

118. A ship in latitude 44° 49′ S., longitude 119° 42′ E., sails due west until it reaches longitude 117° 16′ E. Find the distance made.

In **Middle Latitude Sailing** the departure between two places is measured on that parallel of latitude which lies midway between the parallels of the two places. Except in very high latitudes or excessive runs, this assumption produces no great error. Hence, in middle latitude sailing,

Diff. long. = depart. × sec mid. lat.

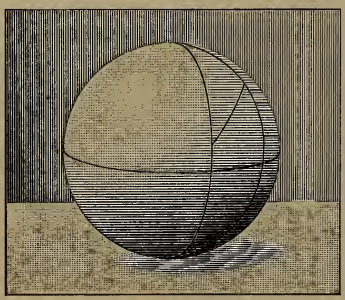


FIG. 71

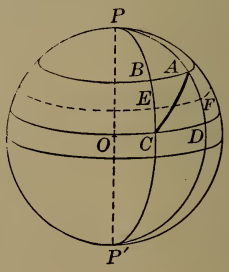


FIG. 72

119. A ship leaves latitude 31° 14′ N., longitude 42° 19′ W., and sails E.N.E. 325 miles. Find the position reached.

120. Find the bearing and distance of Cape Cod from Havana. (Cape Cod 42° 2′ N., 70° 3′ W.; Havana, 23° 9′ N., 82° 22′ W.)

121. Leaving latitude 49° 57′ N., longitude 15° 16′ W., a ship sails between S. and W. till the departure is 194 miles, and the latitude is 47° 18′ N. Find the course, distance, and longitude reached.

122. Leaving latitude 42° 30′ N., longitude 58° 51′ W., a ship sails S.E. by S. 300 miles. Find the position reached.

123. Leaving latitude 49° 57′ N., longitude 30° W., a ship sails S. 39° W., and reaches latitude 47° 44′ N. Find the distance, and longitude reached.

124. Leaving latitude 37° N., longitude 32° 16′ W., a ship sails between N. and W. 300 miles, and reaches latitude 41° N. Find the course, and longitude reached.

125. Leaving latitude 50° 10′ S., longitude 30° E., a ship sails E.S.E., making a departure of 160 miles. Find the distance, and position reached.

126. Leaving latitude 49° 30′ N., longitude 25° W., a ship sails between S. and E. 215 miles, making a departure of 167 miles. Find the course, and position reached.

127. Leaving latitude 43° S., longitude 21° W., a ship sails 273 miles, and reaches latitude 40° 17′ S. What are the *two* courses and longitudes which will satisfy the data?

128. Leaving latitude 17° N., longitude 119° E., a ship sails 219 miles, making a departure of 162 miles. What four sets of answers do we get?

129. A ship in latitude 30° sails due east 360 statute miles. What is the shortest distance from the point left to the point reached? Solve the same problem for latitude 45°, 60°.

TRAVERSE SAILING

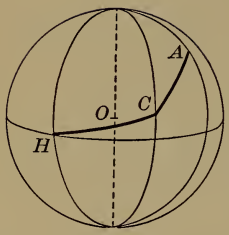


FIG. 73

Traverse Sailing is the application of the principles of Plane and Middle Latitude Sailing to cases when the ship sails from one point to another on two or more different courses. Each course is worked by itself, and these independent results are combined, as may be seen in the solution of the following examples.

130. Leaving latitude 37° 16′ S., longitude 18° 42′ W., a ship sails N.E. 104 miles, then N.N.W. 60 miles, then W. by S. 216 miles. Find the position reached, and its bearing and distance from the point left.

We have, for the first course, difference of latitude 73.5 N., departure 73.5 E.; for the second course, difference of latitude 55.4 N., departure 23 W.; for the third course, difference of latitude 42.1 S., departure 211.8 W.

On the whole, then, the ship has made 128.9 miles of north latitude, and 42.1 miles of south latitude. The place reached is therefore on a parallel of latitude 86.8 miles to the north of the parallel left, that is, in latitude 35° 49.2′ S.

The departure is, in the same way, found to be 161.3 miles W.; and the middle latitude is 36° 32.6′. With these data, and the formula after Example 118, we find the difference of longitude to be 201′, or 3° 21′ W. Hence, the longitude reached is 22° 3′ W.

With the difference of latitude 86.8 miles, and the departure 161.3 miles, we find the course to be N. 61° 43′ W., and the distance 183.2 miles. The ship has reached the same point that it would have reached if it had sailed directly on a course N. 61° 43′ W. for a distance of 183.2 miles.

131. A ship leaves Cape Cod (Example 120), and sails S.E. by S. 114 miles, N. by E. 94 miles, W.N.W. 42 miles. Solve as in Example 130.

132. A ship leaves Cape of Good Hope (latitude 34° 22′ S., longitude 18° 30′ E.), and sails N.W. 126 miles, N. by E. 84 miles, W.S.W. 217 miles. Solve as in Example 130.

EXERCISE XXIV

PROBLEMS IN GONIOMETRY

Prove that:

1. $\sin x + \cos x = \sqrt{2} \cos (x - \frac{1}{4}\pi)$.
2. $\sin x - \cos x = -\sqrt{2} \cos (x + \frac{1}{4}\pi)$.
3. $\sin x + \sqrt{3} \cos x = 2 \sin (x + \frac{1}{3}\pi)$.
4. $\sin (x + \frac{1}{3}\pi) + \sin (x - \frac{1}{3}\pi) = \sin x$.
5. $\cos (x + \frac{1}{6}\pi) + \cos (x - \frac{1}{6}\pi) = \sqrt{3} \cos x$.
6. $\tan x + \sec x = \tan (\frac{1}{2}x + \frac{1}{4}\pi)$.
7. $\tan x + \sec x = \dfrac{1}{\sec x - \tan x}$.
8. $\dfrac{1 - \tan x}{1 + \tan x} = \dfrac{\cot x - 1}{\cot x + 1}$.
9. $\dfrac{\sin x}{1 + \cos x} + \dfrac{1 + \cos x}{\sin x} = 2 \csc x$.
10. $\tan x + \cot x = 2 \csc 2x$.
11. $\cot x - \tan x = 2 \cot 2x$.
12. $1 + \tan x \tan 2x = \sec 2x$.
13. $\sec 2x = \dfrac{\sec^2 x}{2 - \sec^2 x}$.
14. $2 \sec 2x = \sec (x + 45°) \sec (x - 45°)$.

15. $\tan 2x + \sec 2x = \dfrac{\cos x + \sin x}{\cos x - \sin x}$.

16. $\sin 2x = \dfrac{2 \tan x}{1 + \tan^2 x}$.

17. $2 \sin x + \sin 2x = \dfrac{2 \sin^3 x}{1 - \cos x}$.

18. $\sin 3x = \dfrac{\sin^2 2x - \sin^2 x}{\sin x}$.

19. $\tan 3x = \dfrac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$.

20. $\dfrac{\tan 2x + \tan x}{\tan 2x - \tan x} = \dfrac{\sin 3x}{\sin x}$.

21. $\sin (x + y) + \cos (x - y) = 2 \sin (x + \frac{1}{4}\pi) \sin (y + \frac{1}{4}\pi)$.

22. $\sin (x + y) - \cos (x - y) = -2 \sin (x - \frac{1}{4}\pi) \sin (y - \frac{1}{4}\pi)$.

23. $\tan x + \tan y = \dfrac{\sin (x + y)}{\cos x \cos y}$.

24. $\tan (x + y) = \dfrac{\sin 2x + \sin 2y}{\cos 2x + \cos 2y}$.

25. $\dfrac{\sin x + \cos y}{\sin x - \cos y} = \dfrac{\tan \{\frac{1}{2}(x + y) + 45°\}}{\tan \{\frac{1}{2}(x - y) - 45°\}}$.

26. $\sin 2x + \sin 4x = 2 \sin 3x \cos x$.

27. $\sin 4x = 4 \sin x \cos x - 8 \sin^3 x \cos x$
$= 8 \cos^3 x \sin x - 4 \cos x \sin x$.

28. $\cos 4x = 1 - 8 \cos^2 x + 8 \cos^4 x = 1 - 8 \sin^2 x + 8 \sin^4 x$.

29. $\cos 2x + \cos 4x = 2 \cos 3x \cos x$.

30. $\sin 3x - \sin x = 2 \cos 2x \sin x$.

31. $\sin^3 x \sin 3x + \cos^3 x \cos 3x = \cos^3 2x$.

32. $\cos^4 x - \sin^4 x = \cos 2x$.

33. $\cos^4 x + \sin^4 x = 1 - \frac{1}{2}\sin^2 2x.$

34. $\cos^6 x - \sin^6 x = (1 - \sin^2 x \cos^2 x)\cos 2x.$

35. $\cos^6 x + \sin^6 x = 1 - 3\sin^2 x \cos^2 x.$

36. $\dfrac{\sin 3x + \sin 5x}{\cos 3x - \cos 5x} = \cot x.$

37. $\dfrac{\sin 3x + \sin 5x}{\sin x + \sin 3x} = 2\cos 2x.$

38. $\csc x - 2\cot 2x \cos x = 2\sin x.$

39. $(\sin 2x - \sin 2y)\tan(x + y) = 2(\sin^2 x - \sin^2 y).$

40. $(1 + \cot x + \tan x)(\sin x - \cos x) = \dfrac{\sec x}{\csc^2 x} - \dfrac{\csc x}{\sec^2 x}$

41. $\sin x + \sin 3x + \sin 5x = \dfrac{\sin^2 3x}{\sin x}.$

42. $\dfrac{3\cos x + \cos 3x}{3\sin x - \sin 3x} = \cot^3 x.$

43. $\sin 3x = 4\sin x \sin(60° + x)\sin(60° - x).$

44. $\sin 4x = 2\sin x \cos 3x + \sin 2x.$

45. $\sin x + \sin(x - \frac{2}{3}\pi) + \sin(\frac{1}{3}\pi - x) = 0.$

46. $\cos x \sin(y - z) + \cos y \sin(z - x) + \cos z \sin(x - y) = 0.$

47. $\cos(x + y)\sin y - \cos(x + z)\sin z$
$= \sin(x + y)\cos y - \sin(x + z)\cos z.$

48. $\cos(x + y + z) + \cos(x + y - z) + \cos(x - y + z)$
$+ \cos(y + z - x) = 4\cos x \cos y \cos z.$

49. $\sin(x + y)\cos(x - y) + \sin(y + z)\cos(y - z)$
$+ \sin(z + x)\cos(z - x) = \sin 2x + \sin 2y + \sin 2z.$

50. $\dfrac{\sin 75° + \sin 15°}{\sin 75° - \sin 15°} = \tan 60°.$

51. $\cos 20° + \cos 100° + \cos 140° = 0.$

52. $\cos 36° + \sin 36° = \sqrt{2} \cos 9°$.

53. $\tan 11° 15' + 2 \tan 22° 30' + 4 \tan 45° = \cot 11° 15'$.

If A, B, C are the angles of a plane triangle, prove that:

54. $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$.

55. $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$.

56. $\sin 3A + \sin 3B + \sin 3C = -4 \cos \frac{3A}{2} \cos \frac{3B}{2} \cos \frac{3C}{2}$.

57. $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C$.

If $A + B + C = 90°$, prove that:

58. $\tan A \tan B + \tan B \tan C + \tan C \tan A = 1$.

59. $\sin^2 A + \sin^2 B + \sin^2 C = 1 - 2 \sin A \sin B \sin C$.

60. $\sin 2A + \sin 2B + \sin 2C = 4 \cos A \cos B \cos C$.

Prove that:

61. $\sin(\sin^{-1} x + \sin^{-1} y) = x\sqrt{1 - y^2} + y\sqrt{1 - x^2}$.

62. $\tan(\tan^{-1} x + \tan^{-1} y) = \frac{x + y}{1 - xy}$.

63. $2 \tan^{-1} x = \tan^{-1} \frac{2x}{1 - x^2}$.

64. $2 \sin^{-1} x = \sin^{-1}(2x\sqrt{1 - x^2})$.

65. $2 \cos^{-1} x = \cos^{-1}(2x^2 - 1)$.

66. $3 \tan^{-1} x = \tan^{-1} \frac{3x - x^3}{1 - 3x^2}$.

67. $\sin^{-1} \sqrt{\frac{x}{y}} = \tan^{-1} \sqrt{\frac{x}{y - x}}$.

68. $\sin^{-1} \sqrt{\frac{x - y}{x - z}} = \tan^{-1} \sqrt{\frac{x - y}{y - z}}$.

69. $\sin^{-1}x = \sec^{-1}\dfrac{1}{\sqrt{1-x^2}}$.

70. $2\sec^{-1}x = \tan^{-1}\dfrac{2\sqrt{x^2-1}}{2-x^2}$.

71. $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = 45°$.

72. $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} = \tan^{-1}\frac{4}{7}$.

73. $\sin^{-1}\frac{3}{5} + \sin^{-1}1\frac{2}{13} = \sin^{-1}\frac{63}{65}$.

74. $\sin^{-1}\dfrac{1}{\sqrt{82}} + \sin^{-1}\dfrac{4}{\sqrt{41}} = 45°$.

75. $\sec^{-1}\frac{5}{3} + \sec^{-1}1\frac{3}{2} = 75°\ 45'$.

76. $\tan^{-1}(2+\sqrt{3}) - \tan^{-1}(2-\sqrt{3}) = \sec^{-1}2$.

77. $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} = 45°$.

78. $\tan^{-1}\dfrac{1}{1-2x+4x^2} + \tan^{-1}\dfrac{1}{1+2x+4x^2} = \tan^{-1}\dfrac{1}{2x^2}$.

79. Given $\cos x = \frac{3}{5}$; find $\sin\frac{1}{2}x$ and $\cos\frac{1}{2}x$.

80. Given $\tan x = \frac{1}{2}$; find $\tan\frac{1}{2}x$.

81. Given $\sin x + \cos x = \sqrt{\frac{1}{2}}$; find $\cos 2x$.

82. Given $\tan 2x = \frac{24}{7}$; find $\sin x$.

83. Given $\cos 3x = \frac{23}{27}$; find $\tan x$.

84. Given $2\csc x - \cot x = \sqrt{3}$; find $\sin\frac{1}{2}x$.

85. Find $\sin 18°$ and $\cos 36°$.

Find the value of:

86. $a\sec x + b\csc x$, when $\tan x = \sqrt[3]{\dfrac{b}{a}}$.

87. $\sin 3x$, when $\sin 2x = \sqrt{1-m^2}$.

88. $\sin x$, when $\tan^2 x + 3\cot^2 x = 4$.

89. $\dfrac{\csc^2 x - \sec^2 x}{\csc^2 x + \sec^2 x}$, when $\tan x = \sqrt{\frac{1}{7}}$.

90. $\cos x$, when $5 \tan x + \sec x = 5$.

91. $\sec x$, when $\tan x = \dfrac{a}{\sqrt{2a+1}}$.

Simplify the following expressions:

92. $\dfrac{(\cos x + \cos y)^2 + (\sin x + \sin y)^2}{\cos^2 \frac{1}{2}(x - y)}$.

93. $\dfrac{\sin(x + 2y) - 2\sin(x + y) + \sin x}{\cos(x + 2y) - 2\cos(x + y) + \cos x}$.

94. $\dfrac{\sin(x - z) + 2\sin x + \sin(x + z)}{\sin(y - z) + 2\sin y + \sin(y + z)}$.

95. $\dfrac{\cos 6x - \cos 4x}{\sin 6x + \sin 4x}$.

96. $\tan^{-1}(2x + 1) + \tan^{-1}(2x - 1)$.

97. $\dfrac{1}{1 + \sin^2 x} + \dfrac{1}{1 + \cos^2 x} + \dfrac{1}{1 + \sec^2 x} + \dfrac{1}{1 + \csc^2 x}$.

98. $2\sec^2 x - \sec^4 x - 2\csc^2 x + \csc^4 x$.

SOLUTION OF SINGLE EQUATIONS

To **solve** a single equation that involves different functions of the same angle, or the same or different functions of related angles, first transform the equation, if necessary, into an equivalent equation that involves a single function of the same angle.

Employ the method of factoring, if possible, in the algebraic part of the solution.

Completely solve each equation, and check the results by substitution in the given equation.

Solve $\cos x = \sin 2x$.

By [12], p. 58, $\sin 2x = 2 \sin x \cos x$.

$\therefore \cos x = 2 \sin x \cos x$.

$\therefore (1 - 2 \sin x) \cos x = 0$.

$\therefore \cos x = 0$, or $1 - 2 \sin x = 0$.

$\therefore x = 90°$ or $270°$, or $30°$ or $150°$.

Each of these values satisfies the given equation.

Solve the following equations:

99. $\sin x = 2 \sin (\frac{1}{3}\pi + x)$.

100. $\sin 2x = 2 \cos x$.

101. $\cos 2x = 2 \sin x$.

102. $\sin x + \cos x = 1$.

103. $\sin x + \cos 2x = 4 \sin^2 x$.

104. $4 \cos 2x + 3 \cos x = 1$.

105. $\sin x + \sin 2x = \sin 3x$.

106. $\sin 2x = 3 \sin^2 x - \cos^2 x$.

107. $\cot \theta = \frac{1}{3} \tan \theta$.

108. $2 \sin \theta = \cos \theta$.

109. $2 \sin^2 x + 5 \sin x = 3$.

110. $\tan x \sec x = \sqrt{2}$.

111. $\sin x = \cos 2x$.

112. $\tan x \tan 2x = 2$.

113. $\sec x = 4 \csc x$.

114. $\cos \theta + \cos 2\theta = 0$.

115. $\cot \frac{1}{2}\theta + \csc \theta = 2$.

116. $\cot x \tan 2x = 3$.

117. $\sin x \sec 2x = 1$.

118. $\sin^2 x + \sin 2x = 1$.

119. $\cos x \sin 2x \csc x = 1$.

120. $\cot x \tan 2x = \sec 2x$.

121. $\sin 2x = \cos 4x$.

122. $\sin 2z \cot z - \sin^2 z = \frac{1}{2}$

123. $\tan x + \tan 2x = \tan 3x$.

124. $\cot x - \tan x = \sin x + \cos x$.

125. $\tan^2 x = \sin 2x$.

126. $\tan x + \cot x = \tan 2x$.

127. $\dfrac{1 - \tan x}{1 + \tan x} = \cos 2x$.

128. $\sin x + \sin 2x = 1 - \cos 2x.$

129. $\sec 2x + 1 = 2 \cos x.$

130. $\tan 2x + \tan 3x = 0.$

131. $\tan(\frac{1}{4}\pi + x) + \tan(\frac{1}{4}\pi - x) = 4.$

132. $\sqrt{1 + \sin x} - \sqrt{1 - \sin x} = 2 \cos x.$

133. $\tan x \tan 3x = -\frac{2}{5}.$

134. $\sin(45° + x) + \cos(45° - x) = 1.$

135. $\tan x + \sec x = a.$

136. $\cos 2x = a(1 - \cos x).$

137. $(1 - \tan x)\cos 2x = a(1 + \tan x).$

138. $\sin^6 x + \cos^6 x = \frac{7}{12} \sin^2 2x.$

139. $\cos 3x + 8 \cos^3 x = 0.$

140. $\sec(x + 120°) + \sec(x - 120°) = 2 \cos x.$

141. $\csc x = \cot x + \sqrt{3}.$

142. $4 \cos 2x + 6 \sin x = 5$

143. $\cos x - \cos 2x = 1.$

144. $\sin 4x - \sin 2x = \sin x.$

145. $2 \sin^2 x + \sin^2 2x = 2.$

146. $\cos 5x + \cos 3x + \cos x = 0.$

147. $\sec x - \cot x = \csc x - \tan x.$

148. $\tan^2 x + \cot^2 x = \frac{10}{3}.$

149. $\sin 4x - \cos 3x = \sin 2x.$

150. $\sin x + \cos x = \sec x.$

151. $2 \cos x \cos 3x + 1 = 0.$

152. $\cos 3x - 2 \cos 2x + \cos x = 0.$

153. $\tan 2x \tan x = 1.$

154. $\sin(x + 12°) + \sin(x - 8°) = \sin 20°.$

155. $\tan(60° + x)\tan(60° - x) = -2.$

156. $\sin(x + 120°) + \sin(x + 60°) = \frac{3}{2}.$

157. $\sin(x + 30°)\sin(x - 30°) = \frac{1}{2}.$

158. $\sin^4 x + \cos^4 x = \frac{5}{8}.$

159. $\sin^4 x - \cos^4 x = \frac{7}{25}.$

160. $\tan(x + 30°) = 2\cos x.$

161. $\sec x = 2\tan x + \frac{1}{4}.$

162. $\sin 11x \sin 4x + \sin 5x \sin 2x = 0.$

163. $\cos x + \cos 3x + \cos 5x + \cos 7x = 0.$

164. $\sin(x + 12°)\cos(x - 12°) = \cos 33° \sin 57°.$

165. $\sin^{-1} x + \sin^{-1} \frac{1}{2} x = 120°.$

166. $\tan^{-1} x + \tan^{-1} 2x = \tan^{-1} 3\sqrt{3}.$

167. $\sin^{-1} x + 2\cos^{-1} x = \frac{2}{3}\pi.$

168. $\sin^{-1} x + 3\cos^{-1} x = 210°.$

169. $\tan^{-1} x + 2\cot^{-1} x = 135°.$

170. $\tan^{-1}(x + 1) + \tan^{-1}(x - 1) = \tan^{-1} 2x.$

171. $\tan^{-1}\frac{x+2}{x+1} + \tan^{-1}\frac{x-2}{x-1} = \frac{3}{4}\pi.$

172. $\tan^{-1}\frac{2x}{1-x^2} = 60°.$

173. $\cos 2\theta \sec\theta + \sec\theta + 1 = 0.$

174. $\sin x \cos 2x \tan x \cot 2x \sec x \csc 2x = 1.$

175. $\sin\frac{1}{2}x(\cos 2x - 2)(1 - \tan^2 x) = 0.$

HINT. Equate to 0 each factor except the second. The second factor cannot equal 0.

176. $\sin 3x = \cos 2x - 1$.

177. $\tan x + \tan 2x = 0$.

178. $\sin 2\theta = \cos 3\theta$.

179. $(3 - 4\cos^2 x)\sin 2x = 0$.

180. $\sin x + \sin 2x + \sin 3x = 0$.

181. $\sin\theta + 2\sin 2\theta + 3\sin 3\theta = 0$.

182. $\sin^2 x \cos^2 x - \cos^2 x - \sin^2 x + 1 = 0$.

183. $\sin x + \sin 3x = \cos x - \cos 3x$.

184. $(1 - \sqrt{1 - \tan^2 x})\cos 2x \operatorname{vers} 3x = 0$.

185. $\tan(\theta + 45°) = 8\tan\theta$.

186. $\sin(x - 30°) = \frac{1}{2}\sqrt{3}\sin x$.

187. $\tan(\theta + 45°)\tan\theta = 2$.

188. $\sin^{-1}\frac{1}{2}x = 30°$.

SYSTEMS OF EQUATIONS

189. Solve for x and y the system

$$x\sin\alpha + y\sin\beta = a, \tag{1}$$

$$x\cos\alpha + y\cos\beta = b. \tag{2}$$

(1) $\times \cos\alpha$, $x\sin\alpha\cos\alpha + y\sin\beta\cos\alpha = a\cos\alpha$. (3)

(2) $\times \sin\alpha$, $x\sin\alpha\cos\alpha + y\cos\beta\sin\alpha = b\sin\alpha$. (4)

(3) $-$ (4), $y(\sin\beta\cos\alpha - \cos\beta\sin\alpha) = a\cos\alpha - b\sin\alpha$. (5)

$$\therefore\ y = \frac{a\cos\alpha - b\sin\alpha}{\sin(\beta - \alpha)}.$$

Similarly,
$$x = \frac{b\sin\beta - a\cos\beta}{\sin(\beta - \alpha)}.$$

190. Solve for x and y the system

$$\sin x + \sin y = a, \tag{1}$$

$$\cos x + \cos y = b. \tag{2}$$

Transform (1) and (2), by Sect. XXXII,

by [20], p. 59, $2 \sin \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y) = a,$ (3)

by [22], p. 59, $2 \cos \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y) = b.$ (4)

(3) ÷ (4), $\tan \frac{1}{2}(x+y) = \frac{a}{b}.$ (5)

$$\therefore \sin \tfrac{1}{2}(x+y) = \frac{a}{\sqrt{a^2+b^2}}. \quad (6)$$

Substitute value of $\sin \frac{1}{2}(x+y)$ in (3),

$$\cos \tfrac{1}{2}(x-y) = \tfrac{1}{2}\sqrt{a^2+b^2}. \quad (7)$$

From (5), $x + y = 2 \tan^{-1}\frac{a}{b}.$ (8)

From (7), $x - y = 2 \cos^{-1}\tfrac{1}{2}\sqrt{a^2+b^2}.$ (9)

Whence $x = \tan^{-1}\frac{a}{b} + \cos^{-1}\tfrac{1}{2}\sqrt{a^2+b^2},$

and $y = \tan^{-1}\frac{a}{b} - \cos^{-1}\tfrac{1}{2}\sqrt{a^2+b^2}.$

191. Solve for r and θ the system

$$r \sin \theta = a, \quad (1)$$

$$r \cos \theta = b. \quad (2)$$

(1) ÷ (2), $\tan \theta = \frac{a}{b}.$ (3)

From (3), $\theta = \tan^{-1}\frac{a}{b}.$ (4)

Square (1) and (2) and add,

$$r^2(\sin^2\theta + \cos^2\theta) = a^2 + b^2.$$

$$\therefore r = \sqrt{a^2+b^2}.$$

192. Solve for r and θ the system

$$r \sin(\theta + \alpha) = a, \quad (1)$$

$$r \cos(\theta + \beta) = b. \quad (2)$$

Expand (1) and (2),

$$r \sin\theta \cos\alpha + r \cos\theta \sin\alpha = a. \quad (3)$$

$$r \cos\theta \cos\beta - r \sin\theta \sin\beta = b. \quad (4)$$

Now solve (3) and (4) for $r \sin\theta$ and $r \cos\theta$, as in Example 189. Then solve for r and θ, as in Example 191.

193. Solve for r, θ, and ϕ the system

$$r \cos \phi \sin \theta = a, \tag{1}$$

$$r \cos \phi \cos \theta = b, \tag{2}$$

$$r \sin \phi = c. \tag{3}$$

(1) $\div$ (2), $\quad \tan \theta = \frac{a}{b}. \quad \therefore \theta = \tan^{-1} \frac{a}{b}. \tag{4}$

Square (1) and (2) and add,

$$r^2 \cos^2 \phi = a^2 + b^2. \tag{5}$$

(3) $\div$ (5), $\quad \tan \phi = \frac{c}{\sqrt{a^2 + b^2}}. \quad \therefore \phi = \tan^{-1} \frac{c}{\sqrt{a^2 + b^2}}. \tag{6}$

Square (3) and add to (5),

$$r^2 = a^2 + b^2 + c^2.$$

$$\therefore r = \sqrt{a^2 + b^2 + c^2}. \tag{7}$$

Solve the following systems for r, θ, ϕ, x, and y:

194. $x \sin 21° + y \cos 44° = 179.70,$
$x \cos 21° + y \sin 44° = 232.30.$

195. $\sin x - \sin y = 0.7038,$
$\cos x - \cos y = -0.7245.$

196. $r \sin \theta = 92.344,$
$r \cos \theta = 205.309.$

197. $r \sin (\theta - 19° 18') = 59.4034,$
$r \cos (\theta - 30° 54') = 147.9347.$

198. $r \cos \phi \cos \theta = -46.7654,$
$r \sin \phi \cos \theta = 81,$
$r \sin \theta = -54.$

199. Eliminate θ from the system

$$x = r(\theta - \sin \theta),$$
$$y = r(1 - \cos \theta).$$

HINT. $1 - \cos \theta = \text{vers } \theta. \quad \therefore \theta = \text{vers}^{-1} \frac{y}{r}.$

CHAPTER VI

CONSTRUCTION OF TABLES

SECTION XLIII

LOGARITHMS

Properties of Logarithms. Any positive number except unity being selected as a *base*, the index or exponent which the base must have to produce a given number is the logarithm of that number to the given base.

Thus, if $a^n = N$, then $n = \log_a N$.

$n = \log_a N$ is read, n is equal to log N to the base a.

Let a be the base, M and N any positive numbers, m and n their logarithms to the base a; so that

$$a^m = M, \qquad a^n = N,$$
$$m = \log_a M, \qquad n = \log_a N.$$

Then, in any system of logarithms:

1. *The logarithm of* 1 *is* 0.

For, $a^0 = 1$. $\therefore 0 = \log_a 1$.

2. *The logarithm of the base itself is* 1.

For, $a^1 = a$. $\therefore 1 = \log_a a$.

3. *The logarithm of the reciprocal of a positive number is the negative of the logarithm of the number.*

For, if $a^n = N$, then $\frac{1}{N} = \frac{1}{a^n} = a^{-n}$.

$$\therefore \log_a\left(\frac{1}{N}\right) = -n = -\log_a N.$$

4. *The logarithm of the product of two or more positive numbers is found by adding together the logarithms of the several factors.*

For, $$M \times N = a^m \times a^n = a^{m+n}.$$

$$\therefore \log_a (M \times N) = m + n = \log_a M + \log_a N.$$

Similarly for the product of three or more factors.

5. *The logarithm of the quotient of two positive numbers is found by subtracting the logarithm of the divisor from the logarithm of the dividend.*

For, $$\frac{M}{N} = \frac{a^m}{a^n} = a^{m-n}.$$

$$\therefore \log_a \left(\frac{M}{N}\right) = m - n = \log_a M - \log_a N.$$

6. *The logarithm of a power of a positive number is found by multiplying the logarithm of the number by the exponent of the power.*

For, $$N^p = (a^n)^p = a^{np}.$$

$$\therefore \log_a (N^p) = np = p \log_a N.$$

7. *The logarithm of the real positive value of a root of a positive number is found by dividing the logarithm of the number by the index of the root.*

For, $$\sqrt[r]{N} = \sqrt[r]{a^n} = a^{\frac{n}{r}}.$$

$$\therefore \log_a \sqrt[r]{N} = \frac{n}{r} = \frac{\log_a N}{r}.$$

Change of System. Logarithms to any base a may be converted into logarithms to any other base b as follows:

Let N be any number, and let

$$n = \log_a N \text{ and } m = \log_b N.$$

Then, $$N = a^n \text{ and } N = b^m.$$

$$\therefore a^n = b^m.$$

Taking logarithms to any base whatever,

$$n \log a = m \log b,$$

or,
$$\log a \times \log_a N = \log b \times \log_b N,$$

from which $\log_b N$ may be found when $\log a$, $\log b$, and $\log_a N$ are given; and conversely, $\log_a N$ may be found when $\log a$, $\log b$, and $\log_b N$ are given.

Two Important Systems. Although the number of different systems of logarithms is unlimited, there are but two systems which are in common use. These are:

1. The common system, also called the Briggs, denary, or decimal system, of which the base is 10.

2. The natural system of which the base is the fixed value. which the sum of the series

$$1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \cdots$$

approaches as the number of terms is indefinitely increased. This fixed value, correct to seven places of decimals, is 2.7182818, and is denoted by the letter e.

The common system is used in actual calculation; the natural system is used in the higher mathematics.

EXERCISE XXV

1. Given $\log_{10} 2 = 0.30103$, $\log_{10} 3 = 0.47712$, $\log_{10} 7 = 0.84510$; find $\log_{10} 6$, $\log_{10} 14$, $\log_{10} 21$, $\log_{10} 4$, $\log_{10} 12$, $\log_{10} 5$, $\log_{10} \frac{1}{2}$, $\log_{10} \frac{1}{4}$, $\log_{10} \frac{7}{9}$, $\log_{10} \frac{21}{20}$.

2. With the data of Example 1, find $\log_2 10$, $\log_2 5$, $\log_3 5$, $\log_7 \frac{1}{2}$, $\log_5 3\frac{9}{43}$.

3. Given $\log_{10} e = 0.43429$; find $\log_e 2$, $\log_e 3$, $\log_e 5$, $\log_e 7$, $\log_e 8$, $\log_e 9$, $\log_e \frac{2}{3}$, $\log_e \frac{4}{5}$, $\log_e \frac{35}{27}$, $\log_e \frac{7}{60}$.

4. Find x from the equations $5^x = 12$, $16^x = 10$, $27^x = 4$.

SECTION XLIV

EXPONENTIAL AND LOGARITHMIC SERIES

Exponential Series. By the binomial theorem,

$$\left(1+\frac{1}{n}\right)^{nx} = 1 + nx \times \frac{1}{n} + \frac{nx\,(nx-1)}{1\cdot 2} \times \frac{1}{n^2} + \frac{nx\,(nx-1)\,(nx-2)}{1\cdot 2\cdot 3} \times \frac{1}{n^3} + \cdots$$

$$= 1 + x + \frac{x\left(x-\frac{1}{n}\right)}{\lfloor 2} + \frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\lfloor 3} + \cdots \quad (1)$$

This equation is true for all real values of x, since the binomial theorem may be extended to the case of incommensurable exponents (Wentworth's *College Algebra*, § 299); it is, however, true only for values of n numerically greater than 1, since $\frac{1}{n}$ must be numerically less than 1 (*College Algebra*, § 418).

As (1) is true for all values of x, it is true when $x = 1$.

$$\therefore \left(1+\frac{1}{n}\right)^{n} = 1 + 1 + \frac{1-\frac{1}{n}}{\lfloor 2} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\lfloor 3} + \cdots \quad (2)$$

But $$\left[\left(1+\frac{1^n}{n}\right)\right]^{x} = \left(1+\frac{1}{n}\right)^{nx}.$$

Hence, from (1) and (2),

$$\left[1 + 1 + \frac{1-\frac{1}{n}}{\lfloor 2} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\lfloor 3} + \cdots\right]^{x}$$

$$= 1 + x + \frac{x\left(x-\frac{1}{n}\right)}{\lfloor 2} + \frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\lfloor 3} + \cdots$$

This last equation is true for all values of n numerically greater than 1. Taking the limits of the two members as n increases without limit, we obtain

$$\left(1+1+\frac{1}{\underline{2}}+\frac{1}{\underline{3}}+\cdots\right)^x = 1+x+\frac{x^2}{\underline{2}}+\frac{x^3}{\underline{3}}+\cdots, \quad (3)$$

and this is true for all values of x. It is easily seen that both series are convergent for all values of x.

The sum of the infinite series in parenthesis is the natural base e.

Hence, by (3), $e^x = 1+x+\frac{x^2}{\underline{2}}+\frac{x^3}{\underline{3}}+\cdots$ (4)

To calculate the value of e, we proceed as follows:

	1.000000
2	1.000000
3	0.500000
4	0.166667
5	0.041667
6	0.008333
7	0.001388
8	0.000198
9	0.000025
	0.000003

Adding, $e = 2.71828.$

To ten places, $e = 2.7182818284.$

Limit of $\left(1+\frac{x}{n}\right)^n$. By the binomial theorem,

$$\left(1+\frac{x}{n}\right)^n = 1+n\times\frac{x}{n}+\frac{n(n-1)}{1\cdot 2}\times\frac{x^2}{n^2}$$

$$+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\times\frac{x^3}{n^3}+\cdots$$

$$= 1+x+\frac{1-\frac{1}{n}}{\underline{2}}x^2+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\underline{3}}x^3+\cdots$$

This equation is true for all values of n greater than x (*College Algebra*, § 418). Take the limit as n increases without limit, x remaining finite; then

$$\lim_{n \doteq \infty} \left(1 + \frac{x}{n}\right)^n = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots$$

$$= e^x = \lim_{n \doteq \infty} \left(1 + \frac{1}{n}\right)^{nx}. \qquad (5)$$

Logarithmic Series.

Let $$y = \log_e(1 + x);$$

then $$1 + x = e^y = \lim_{n \doteq \infty} \left(1 + \frac{y}{n}\right)^n.$$

If n is merely a large number, but not infinite,

$$\left(1 + \frac{y}{n}\right)^n = 1 + x + \epsilon,$$

where ϵ is a variable number which approaches the limit 0, when n increases without limit. Hence,

$$1 + \frac{y}{n} = \sqrt{1 + x + \epsilon},$$

$$y = n \sqrt[n]{1 + x + \epsilon} - n.$$

If n increases without limit, and consequently ϵ approaches 0 as a limit, we have

$$y = \lim_{n \doteq \infty} [n \sqrt[n]{1 + x} - n].$$

If x is less than 1, we may expand the right-hand member of this equation by the binomial theorem. The result is

$$y = \lim_{n \doteq \infty} \left[n \left\{ 1 + \frac{1}{n} x + \frac{1}{n}\left(\frac{1}{n} - 1\right)\frac{x^2}{\lfloor 2} + \cdots \right\} - n \right]$$

$$= \lim_{n \doteq \infty} \left[x + \left(\frac{1}{n} - 1\right)\frac{x^2}{\lfloor 2} + \left(\frac{1}{n} - 1\right)\left(\frac{1}{n} - 2\right)\frac{x^3}{\lfloor 3} + \cdots \right]$$

$$= x - \frac{x^2}{\lfloor 2} + \frac{\lfloor 2\, x^3}{\lfloor 3} - \frac{\lfloor 3\, x^4}{\lfloor 4} + \cdots$$

$$\therefore \log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

This series is known as the *logarithmic series.* It is convergent only if x lies between -1 and $+1$, or is equal to $+1$. Even within these limits it converges rather slowly, and for these reasons it is not well adapted to the computation of logarithms. A more convenient series is obtained as follows.

Calculation of Logarithms. The equation

$$\log_e(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots \qquad (1)$$

holds true for all values of y numerically less than 1; therefore, if it holds true for any particular value of y less than 1, it will hold true when we put $-y$ for y; this gives

$$\log_e(1-y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} - \cdots \qquad (2)$$

Subtracting (2) from (1), since

$$\log_e(1+y) - \log_e(1-y) = \log_e\left(\frac{1+y}{1-y}\right),$$

we find

$$\log_e\left(\frac{1+y}{1-y}\right) = 2\left(y + \frac{y^3}{3} + \frac{y^5}{5} + \cdots\right).$$

Put

$$y = \frac{1}{2z+1};$$

then

$$\frac{1+y}{1-y} = \frac{z+1}{z},$$

and

$$\log_e\left(\frac{z+1}{z}\right) = \log_e(z+1) - \log_e z$$

$$= 2\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \cdots\right).$$

This series is convergent for all positive values of z.

Logarithms to any base a can be calculated by the series:

$$\log_a(z+1) - \log_a z = \frac{2}{\log_e a}\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \cdots\right).$$

EXAMPLE. Calculate $\log_e 2$ to five places of decimals.

Let $z = 1$; then $z+1 = 2$, $2z+1 = 3$,

and $$\log_e 2 = \frac{2}{3} + \frac{2}{3 \times 3^3} + \frac{2}{5 \times 3^5} + \frac{2}{7 \times 3^7} + \cdots$$

The work may be arranged as follows:

3	2.000000		
9	0.666667	÷ 1	= 0.666667
9	0.074074	÷ 3	= 0.024691
9	0.008230	÷ 5	= 0.001646
9	0.000914	÷ 7	= 0.000131
9	0.000102	÷ 9	= 0.000011
	0.000011	÷ 11	= 0.000001
		$\log_e 2$	= 0.693147

NOTE. In calculating logarithms the accuracy of the work may be tested every time we come to a composite number by adding the logarithms of the several factors. In fact, the logarithms of composite numbers are best found by addition, and then only the logarithms of prime numbers need be computed by the series.

EXERCISE XXVI

1. Calculate to five places of decimals $\log_e 3$.
2. Calculate to five places of decimals $\log_e 5$.
3. Calculate to five places of decimals $\log_e 7$.
4. Calculate to ten places of decimals $\log_e 10$.
5. Calculate to five places of decimals $\log_{10} 2$, $\log_{10} e$, $\log_{10} 11$.

SECTION XLV

TRIGONOMETRIC FUNCTIONS OF SMALL ANGLES

Let AOP (Fig. 74) be any angle less than 90° and x its circular measure. Describe a circle of unit radius about O as a centre and take $\angle AOP' = -\angle AOP$. Draw the tangents to the circle at P and P', meeting OA in T. Then, from Geometry,

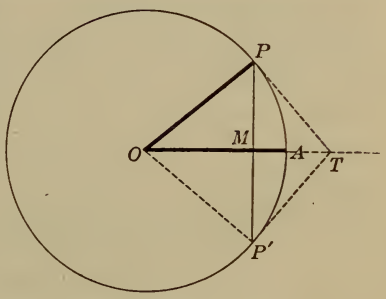


FIG. 74

$$\text{chord } PP' < \text{arc } PP' < PT + P'T,$$

or, by dividing by 2,

$$MP < \text{arc } AP < PT,$$

or

$$\sin x < x < \tan x.$$

Hence, dividing by $\sin x$,

$$1 < \frac{x}{\sin x} < \sec x,$$

$$1 > \frac{\sin x}{x} > \cos x. \qquad (1)$$

Then $\dfrac{\sin x}{x}$ lies between $\cos x$ and 1.

If now the angle x is constantly diminished, $\cos x$ approaches the value 1.

Accordingly, the limit of $\dfrac{\sin x}{x}$, as x approaches 0, is 1. In other words, if x is a very small angle, then $\dfrac{\sin x}{x}$ differs from 1 by a small value ϵ; and this small value ϵ approaches 0 as x approaches 0.

EXAMPLE. To find the sine and cosine of 1′.

If x is the circular measure of 1′,

$$x = \frac{2\pi}{360 \times 60} = \frac{3.14159+}{10800} = 0.00029088+,$$

the next figure in x being 8.

Now $\sin x > 0$ but $< x$; hence, $\sin 1'$ lies between 0 and 0.000290889.

Again, $\cos 1' = \sqrt{1 - \sin^2 1'} > \sqrt{1 - (0.0003)^2} > 0.9999999.$

Hence, $\cos 1' = 0.9999999+.$

But, from (1), $\sin x > x \cos x.$

$$\begin{aligned} \therefore \sin 1' &> 0.000290888 \times 0.9999999 \\ &> 0.000290888\,(1 - 0.0000001) \\ &> 0.000290888 - 0.0000000000290888 \\ &> 0.000290887. \end{aligned}$$

Hence, $\sin 1'$ lies between 0.000290887 and 0.000290889; that is, to eight places of decimals

$$\sin 1' = 0.00029088+,$$

the next figure being 7 or 8.

EXERCISE XXVII

Given $\pi = 3.141592653589$:

1. Compute $\sin 1'$, $\cos 1'$, and $\tan 1'$ to eleven places of decimals.

2. Compute $\sin 2'$ by the same method, and also by the formula $\sin 2x = 2 \sin x \cos x$. Carry the operations to nine places of decimals. Do the two results agree?

3. Compute $\sin 1°$ to four places of decimals.

4. From the formula $\cos x = 1 - 2\sin^2\frac{x}{2}$, show that

$$\cos x > 1 - \frac{x^2}{2}.$$

5. Show by aid of a table of natural sines that $\sin x$ and x agree to four places of decimals for all angles less than $4^\circ\ 40'$.

6. If the values of $\log x$ and $\log \sin x$ agree to five decimal places, find from a table the greatest value x can have.

SECTION XLVI

SIMPSON'S METHOD OF CONSTRUCTING A TRIGONOMETRIC TABLE

By Sect. XXXII, p. 59,

$$\sin(A + B) + \sin(A - B) = 2\sin A\cos B.$$

If we put $A = x + 2y,\ B = y,$

we have $\sin(x + 3y) + \sin(x + y) = 2\sin(x + 2y)\cos y,$

or $\sin(x + 3y) = 2\sin(x + 2y)\cos y - \sin(x + y).$

Similarly, $\cos(x + 3y) = 2\cos(x + 2y)\cos y - \cos(x + y).$ (1)

If $y = 1'$, the last two equations become

$$\sin(x + 3') = 2\sin(x + 2')\cos 1' - \sin(x + 1'),$$
$$\cos(x + 3') = 2\cos(x + 2')\cos 1' - \cos(x + 1').$$

Hence, taking x successively equal to $-1'$, $0'$, $1'$, $2'$, $\cdots$, we obtain

$$\sin 2' = 2\sin 1'\cos 1',$$
$$\sin 3' = 2\sin 2'\cos 1' - \sin 1',$$
$$\sin 4' = 2\sin 3'\cos 1' - \sin 2',$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\cos 2' = 2\cos^2 1' - 1,$$
$$\cos 3' = 2\cos 2'\cos 1' - \cos 1',$$
$$\cos 4' = 2\cos 3'\cos 1' - \cos 2',$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Since $\sin 1'$ and $\cos 1'$ are known, these equations enable us to compute step by step the sine and cosine of any angle. The tangent may then be found in each case as the quotient of the sine divided by the cosine.

This process need be carried only as far as 30°. For

$$\sin(30° + x) + \sin(30° - x) = 2 \sin 30° \cos x = \cos x,$$
$$\cos(30° + x) - \cos(30° - x) = -2 \sin 30° \sin x = -\sin x.$$
$$\therefore \sin(30° + x) = \cos x - \sin(30° - x),$$
$$\cos(30° + x) = -\sin x + \cos(30° - x).$$

Moreover, the sines and cosines need be calculated only to 45°, since

$$\sin(45° + x) = \cos(45° - x),$$
$$\cos(45° + x) = \sin(45° - x).$$

In using this method, the multiplication by $\cos 1'$, which occurs at each step, can be simplified by noting that

$$\cos 1' = 0.9999999 = 1 - 0.0000001.$$

NOTE. Simpson's method is superseded in actual practice by much more rapid and convenient processes in which we employ the expansions of the trigonometric functions in infinite series.

EXERCISE XXVIII

1. Compute the sine and cosine of 6′ to seven decimal places.

In Formula (1) let $y = 1°$. Assuming

$$\sin 1° = 0.017454+, \quad \cos 1° = 0.999848+:$$

2. Compute the sine and cosine of two degrees.
3. Compute the sine and cosine of three degrees.
4. Compute the sine and cosine of four degrees.
5. Compute the sine and cosine of five degrees.

SECTION XLVII

DE MOIVRE'S THEOREM

Expressions of the form

$$\cos x + i \sin x,$$

when $i = \sqrt{-1}$, play an important part in modern analysis.

Given two such expressions,

$$\cos x + i \sin x, \quad \cos y + i \sin y$$

their product is

$$\begin{aligned}(\cos x + i \sin x)(\cos y + i \sin y) &= \cos x \cos y - \sin x \sin y + i(\cos x \sin y + \sin x \cos y) \\ &= \cos(x + y) + i \sin(x + y).\end{aligned}$$

Hence, the product of two expressions of the form

$$\cos x + i \sin x, \ \cos y + i \sin y$$

is an expression of the same form in which x or y is replaced by $x + y$. In other words, the angle which enters into such a product is the sum of the angles of the factors.

If x and y are equal, we have at once, from the preceding,

$$(\cos x + i \sin x)^2 = \cos 2x + i \sin 2x;$$

and again,

$$\begin{aligned}(\cos x + i \sin x)^3 &= (\cos x + i \sin x)^2(\cos x + i \sin x) \\ &= (\cos 2x + i \sin 2x)(\cos x + i \sin x) \\ &= \cos 3x + i \sin 3x.\end{aligned}$$

Similarly,

$$(\cos x + i \sin x)^4 = \cos 4x + i \sin 4x,$$

and in general, if n is a positive integer,

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx. \qquad (1)$$

Hence,

To raise the expression cos x + i *sin* x *to the* n*th power when* n *is a positive integer, we have only to multiply the angle* x *by* n.

Again, if n is a positive integer as before,

$$\left(\cos\frac{x}{n}+i\sin\frac{x}{n}\right)^n=\cos x+i\sin x.$$

$$\therefore(\cos x+i\sin x)^{\frac{1}{n}}=\cos\frac{x}{n}+i\sin\frac{x}{n}.$$

Since, however, x may be increased by any integral multiple of 2π without changing $\cos x+i\sin x$, it follows that all the n expressions,

$$\cos\frac{x}{n}+i\sin\frac{x}{n},\quad\cos\frac{x+2\pi}{n}+i\sin\frac{x+2\pi}{n},$$

$$\cos\frac{x+4\pi}{n}+i\sin\frac{x+4\pi}{n},\cdots,$$

$$\cos\frac{x+(n-1)2\pi}{n}+i\sin\frac{x+(n-1)2\pi}{n},$$

are nth roots of $\cos x+i\sin x$. There are no other roots, since

$$\cos\frac{x+n2\pi}{n}+i\sin\frac{x+n2\pi}{n}$$

$$=\cos\left(\frac{x}{n}+2\pi\right)+i\sin\left(\frac{x}{n}+2\pi\right)=\cos\frac{x}{n}+i\sin\frac{x}{n},$$

and $\cos\dfrac{x+(n+1)2\pi}{n}+i\sin\dfrac{x+(n+1)2\pi}{n}$

$$=\cos\left(\frac{x+2\pi}{n}+2\pi\right)+i\sin\left(\frac{x+2\pi}{n}+2\pi\right)$$

$$=\cos\frac{x+2\pi}{n}+i\sin\frac{x+2\pi}{n},$$

and so on.

Hence, if n is a positive integer,

$$(\cos x + i \sin x)^{\frac{1}{n}}$$

$$= \cos \frac{x + 2k\pi}{n} + i \sin \frac{x + 2k\pi}{n} \quad (k = 0, 1, 2, \cdots n - 1). \quad (2)$$

From (1) and (2) it follows at once that if m and n are positive integers,

$$(\cos x + i \sin x)^{\frac{m}{n}} = \{(\cos x + i \sin x)^{\frac{1}{n}}\}^m$$

$$= \cos \frac{m}{n}(x + 2k\pi) + i \sin \frac{m}{n}(x + 2k\pi)$$

$$(k = 0, 1, 2, \cdots n - 1). \quad (3)$$

Finally, if $-\frac{m}{n}$ is a negative fraction,

$$(\cos x + i \sin x)^{-\frac{m}{n}} = \frac{1}{(\cos x + i \sin x)^{\frac{m}{n}}}.$$

But
$$\frac{1}{\cos x + i \sin x} = \frac{\cos x - i \sin x}{(\cos x + i \sin x)(\cos x - i \sin x)}$$

$$= \frac{\cos x - i \sin x}{\cos^2 x + \sin^2 x}$$

$$= \cos x - i \sin x$$

$$= \cos(-x) + i \sin(-x).$$

Hence, $(\cos x + i \sin x)^{-\frac{m}{n}} = \{\cos(-x) + i \sin(-x)\}^{\frac{m}{n}}$

$$= \cos \frac{m}{n}(-x + 2k\pi) + i \sin \frac{m}{n}(-x + 2k\pi),$$

$$(k = 0, 1, 2, \cdots n - 1)$$

$$= \cos\left\{-\frac{m}{n}(x + 2k\pi)\right\} + i \sin\left\{-\frac{m}{n}(x + 2k\pi)\right\},$$

$$(k = 0, 1, 2, \cdots n - 1). \quad (4)$$

Consequently, if n is a positive or negative integer or fraction,

$$(\cos x + i \sin x)^n = \cos [n(x + 2k\pi)] + i \sin [n(x + 2k\pi)],$$
$$(k = 0, 1, 2, \cdots n - 1). \quad (5)$$

EXAMPLE. Find the three cube roots of -1.

We have $-1 = \cos 180° + i \sin 180°$.

$$\therefore (-1)^{\frac{1}{3}} = \cos \frac{180° + 2k\pi}{3} + i \sin \frac{180° + 2k\pi}{3} (k = 0, 1, 2).$$

For the three cube roots of -1 we find, therefore,

$\cos 60° + i \sin 60°$, $\cos 180° + i \sin 180°$, $\cos 300° + i \sin 300°$,

or $$\frac{1 + i\sqrt{3}}{2}, \quad -1, \quad \frac{1 - i\sqrt{3}}{2}.$$

By aid of De Moivre's Theorem, we may express $\sin n\theta$ and $\cos n\theta$, when n is an integer, in terms of $\sin \theta$ and $\cos \theta$.

Thus, $\cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n$

$$= \cos^n \theta + in \cos^{n-1} \theta \sin \theta + i^2 \frac{n(n-1)}{\lfloor 2} \cos^{n-2} \theta \sin^2 \theta$$
$$+ i^3 \frac{n(n-1)(n-2)}{\lfloor 3} \cos^{n-3} \theta \sin^3 \theta + \cdots$$

Or, since $i^2 = -1$, $i^3 = -i$, $i^4 = +1$, $\cdots$,

$$\cos n\theta + i \sin n\theta = \cos^n \theta + in \cos^{n-1} \theta \sin \theta$$
$$- \frac{n(n-1)}{\lfloor 2} \cos^{n-2} \theta \sin^2 \theta - i \frac{n(n-1)(n-2)}{\lfloor 3} \cos^{n-3} \theta \sin^3 \theta + \cdots$$

Equating now the real parts and the imaginary parts separately, we obtain

$$\cos n\theta = \cos^n \theta - \frac{n(n-1)}{\lfloor 2} \cos^{n-2} \theta \sin^2 \theta$$
$$+ \frac{n(n-1)(n-2)(n-3)}{\lfloor 4} \cos^{n-4} \theta \sin^4 \theta - \cdots,$$

$$\sin n\theta = n\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{\lfloor 3}\cos^{n-3}\theta\sin^3\theta$$
$$+\frac{n(n-1)(n-2)(n-3)(n-4)}{\lfloor 5}\cos^{n-5}\theta\sin^5\theta - \cdots$$

EXERCISE XXIX

1. Find the six 6th roots of -1; of $+1$.
2. Find the three cube roots of i.
3. Find the four 4th roots of $-i$.
4. Express $\sin 4\theta$ and $\cos 4\theta$ in terms of $\sin\theta$ and $\cos\theta$.

SECTION XLVIII

EXPANSION OF SIN X, COS X, AND TAN X IN INFINITE SERIES

Let one radian be denoted simply by 1, and let

$$\cos 1 + i\sin 1 = k.$$

Then $$\cos x + i\sin x = (\cos 1 + i\sin 1)^x = k^x,$$

and, putting $-x$ for x,

$$\cos(-x) + i\sin(-x) = \cos x - i\sin x = k^{-x}.$$

That is, $$\cos x + i\sin x = k^x,$$

and $$\cos x - i\sin x = k^{-x}.$$

By taking the sum and difference of these two equations, and dividing the sum by 2 and the difference by $2i$, we have

$$\cos x = \frac{1}{2}(k^x + k^{-x}), \qquad \sin x = \frac{1}{2i}(k^x - k^{-x}).$$

But $$k^x = (e^{\log k})^x = e^{x\log k}, \qquad k^{-x} = e^{-x\log k},$$

and $$e^{x\log k} = 1 + x\log k + \frac{x^2(\log k)^2}{\lfloor 2} + \frac{x^3(\log k)^3}{\lfloor 3} + \cdots,$$

$$e^{-x\log k} = 1 - x\log k + \frac{x^2(\log k)^2}{\lfloor 2} - \frac{x^3(\log k)^3}{\lfloor 3} + \cdots$$

$$\therefore \cos x = \frac{1}{2}(k^x + k^{-x}) = 1 + \frac{x^2(\log k)^2}{\lfloor 2} + \frac{x^4(\log k)^4}{\lfloor 4} + \cdots,$$

$$\sin x = \frac{1}{i}\left\{ x\log k + \frac{x^3(\log k)^3}{\lfloor 3} + \frac{x^5(\log k)^5}{\lfloor 5} + \cdots \right\}.$$

It only remains to find the value of k, and this can be obtained by dividing the last equation through by x and letting x approach 0 indefinitely.

Then we have

$$\underset{x \doteq 0}{\text{limit}} \left(\frac{\sin x}{x}\right) = \frac{1}{i}\log k.$$

But

$$\underset{x \doteq 0}{\text{limit}} \left(\frac{\sin x}{x}\right) = 1.$$

$$\therefore \log k = i.$$

$$\therefore k = e^i.$$

Therefore, we have

$$\cos x = \frac{1}{2}(e^{xi} + e^{-xi}) = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \cdots,$$

$$\sin x = \frac{1}{2i}(e^{xi} - e^{-xi}) = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} \cdots$$

From the last two series we obtain, by division,

$$\tan x = \frac{\sin x}{\cos x} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} \cdots$$

By the aid of these series the trigonometric functions of any angle are readily calculated.

In the computation it must be remembered that x is the *circular measure* of the given angle.

EXERCISE XXX

Verify by the series just obtained that:

1. $\sin^2 x + \cos^2 x = 1$.
2. $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$.
3. $\sin 2x = 2 \sin x \cos x$.
4. $\cos 2x = 1 - 2\sin^2 x$.
5. Find the series for $\sec x$ as far as the term containing the 6th power of x.
6. Find the series for $x \cot x$, noting that
$$x \cot x = \frac{x}{\sin x} \cos x.$$
7. Calculate $\sin 10°$ and $\cos 10°$ to five places of decimals.
8. Calculate $\tan 15°$ to five places of decimals.
9. From the exponential value of $\cos x$ show that
$$\cos 3x = 4\cos^3 x - 3\cos x.$$
10. From the exponential value of $\sin x$ show that
$$\sin 3x = 3\sin x - 4\sin^3 x.$$

SPHERICAL TRIGONOMETRY

CHAPTER VII

THE RIGHT SPHERICAL TRIANGLE

SECTION XLIX

INTRODUCTION

The object of *Spherical Trigonometry* is to explain the method of solving spherical triangles. To *solve* a spherical triangle is to compute any three of its parts when the other three parts are given.

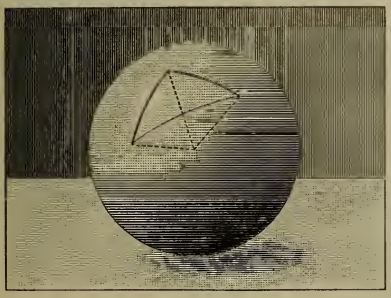


FIG. 75

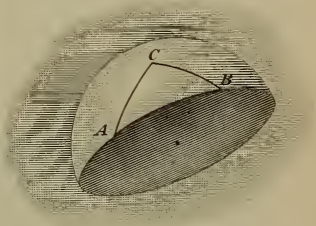


FIG. 76

The sides of a spherical triangle are arcs of great circles. Thus, AB (Fig. 76) is an arc of a great circle. The sides of a spherical triangle are measured in degrees, minutes, and seconds, and therefore by the plane angles formed by radii of

the sphere drawn to the vertices of the triangle. Therefore, the measures of the sides are independent of the length of the radius, which may be assumed to have any convenient numerical value; as, for example, unity.

The angles of a spherical triangle are measured by the dihedral angles made by the planes of the sides. Each angle is also measured by the number of degrees in the arc of a great circle, described from the vertex of the angle as a pole, and included between the sides of the angle.

The sides may have any value from 0° to 360°; but in this work only sides that are less than 180° will be considered. The angles may have any value from 0° to 180°.

A **right spherical triangle** may have one, two, or three right angles.

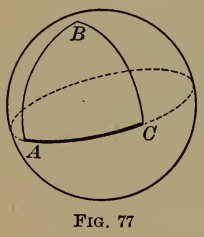
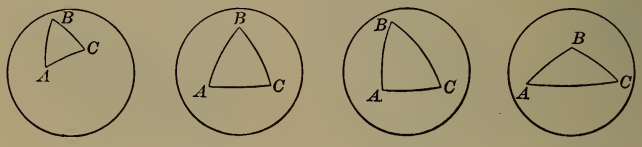


FIG. 77

When a spherical triangle has one or more of its sides equal to a quadrant it is called a **quadrantal triangle** (Fig. 77).

If any two parts of a spherical triangle are either both less than 90° or both greater than 90°, they are said to be **alike in kind**; but if one part is less than 90°, and the other part greater than 90°, they are said to be **unlike in kind.**



Isosceles Equilateral Right Oblique

FIG. 78

Spherical triangles are named isosceles, equilateral, equiangular, right, and oblique, under the same conditions as plane triangles are named isosceles, equilateral, equiangular, right, and oblique.

The following propositions are proved in Geometry. (See Wentworth's *Geometry*, §§ 815, 790, 795, 793.)

1. *If two angles of a spherical triangle are unequal, the sides opposite are unequal, and the greater side is opposite the greater angle; and conversely.*

2. *The sum of the sides of a spherical triangle is less than* 360°.

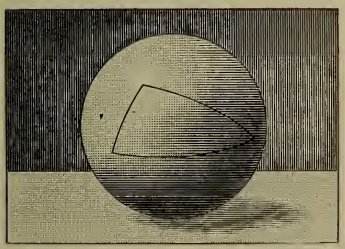


FIG. 79

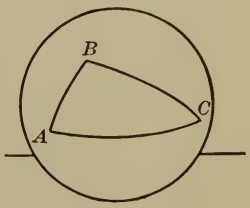


FIG. 80

3. *The sum of the angles of a spherical triangle is greater than* 180° *and less than* 540°.

4. *If, from the vertices of a spherical triangle as poles, arcs of great circles are drawn, another triangle is formed so related to the first that each angle of either triangle is the supplement of the side opposite it in the other triangle.*

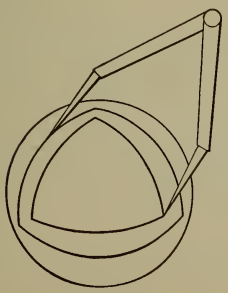


FIG. 81

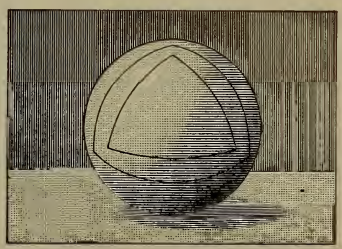


FIG. 82

Two spherical triangles, drawn as explained in Theorem 4, p. 141, are called **polar triangles,** or *supplemental triangles.*

Let A, B, C (Fig. 83) denote the angles of one triangle; a, b, c the sides opposite these angles, respectively; and let A', B', C' and a', b', c' denote the corresponding angles and the corresponding sides of the polar triangle. Then Theorem 4 gives the six following equations:

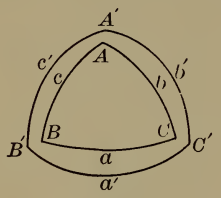


FIG. 83

$$A + a' = 180°,$$
$$B + b' = 180°,$$
$$C + c' = 180°,$$
$$A' + a = 180°,$$
$$B' + b = 180°,$$
$$C' + c = 180°.$$

EXERCISE XXXI

1. The angles of a triangle are 70°, 80°, and 100°. Find the sides of the polar triangle.

2. The sides of a triangle are 40°, 90°, and 125°. Find the angles of the polar triangle.

3. Show that, if a triangle has three right angles, the sides of the triangle are quadrants.

4. Show that, if a triangle has two right angles, the sides opposite these angles are quadrants, and the third angle is measured by the number of degrees in the opposite side.

5. How can the sides of a spherical triangle, measured in degrees, be found in units of length, when the length of the radius of the sphere is known?

6. Find the lengths of the sides of the triangle in Example 2 if the radius of the sphere is 4 feet.

SECTION L

FORMULAS RELATING TO RIGHT SPHERICAL TRIANGLES

As is evident from Examples 3 and 4, Exercise XXXI, the only kind of right spherical triangle that requires further investigation is that which contains *only one* right angle.

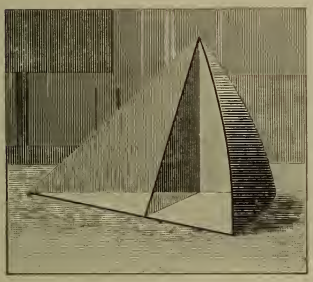


FIG. 84

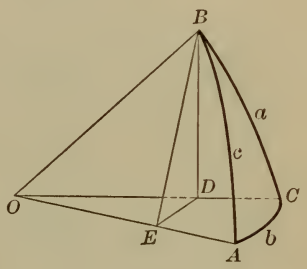


FIG. 85

Let ABC (Fig. 85) be a right spherical triangle having only one right angle; and let A, B, C denote the angles of the triangle; a, b, c, respectively, the opposite sides.

Let C be the right angle; and for the present suppose that each of the other parts is less than 90°, and that the radius of the sphere is 1.

Let planes be passed through the sides, intersecting in the radii OA, OB, and OC.

Also, let a plane $\perp$ to OA be passed through B, cutting OA at E and OC at D. Draw BE, BD, and DE.

BE and DE are each $\perp$ to OA (Wentworth's *Geometry*, § 501); therefore, $\angle BED = A$. The plane BDE is $\perp$ to the plane AOC (Wentworth's *Geometry*, § 554); hence, BD, which is the intersection of the planes BDE and BOC, is $\perp$ to the plane AOC (Wentworth's *Geometry*, § 556), and therefore $\perp$ to OC and DE.

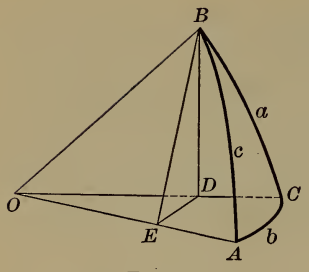


FIG. 86

Now,

$$\cos c = OE = OD \times \cos b,$$

and $OD = \cos a$.

Therefore,

$$\cos c = \cos a \cos b. \qquad [38]$$

Again, $\sin a = BD = BE \times \sin A$,

and $BE = \sin c$.

Therefore,
changing letters,
$$\left.\begin{aligned} \sin a &= \sin c \sin A \\ \sin b &= \sin c \sin B \end{aligned}\right\}. \qquad [39]$$

Again,
$$\cos A = \frac{DE}{BE} = \frac{OE \tan b}{OE \tan c}.$$

Hence,
changing letters,
$$\left.\begin{aligned} \cos A &= \tan b \cot c \\ \cos B &= \tan a \cot c \end{aligned}\right\}. \qquad [40]$$

Again,
$$\cos A = \frac{DE}{BE} = \frac{OD \sin b}{\sin c} = \cos a \frac{\sin b}{\sin c}.$$

By substituting for $\frac{\sin b}{\sin c}$ its value from [39], we obtain

changing letters,
$$\left.\begin{aligned} \cos A &= \cos a \sin B \\ \cos B &= \cos b \sin A \end{aligned}\right\}. \qquad [41]$$

Again,
$$\sin b = \frac{DE}{OD} = \frac{BD \cot A}{OD} = \tan a \cot A.$$

Hence,
changing letters,
$$\left.\begin{aligned} \sin b &= \tan a \cot A \\ \sin a &= \tan b \cot B \end{aligned}\right\}. \qquad [42]$$

If in [38] we substitute for $\cos a$ and $\cos b$ their values from [41], we obtain

$$\cos c = \cot A \cot B. \qquad [43]$$

NOTE. In order to deduce the second formulas in [39]–[42] geometrically, the auxiliary plane must be passed through $A \perp$ to OB.

These ten formulas are sufficient for the solution of any right spherical triangle. In deducing these formulas, all the parts of the triangle, except the right angle, were assumed to be less than 90°. But the formulas hold when this hypothesis is not true.

Let one of the legs a be greater than 90°, and construct a figure for this case (Fig. 88) in the same manner as Fig. 85.

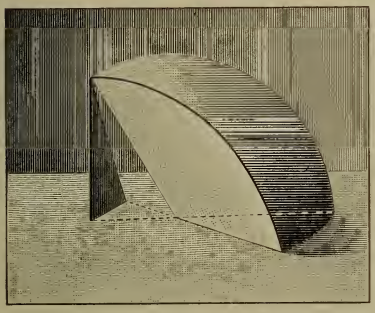


FIG. 87

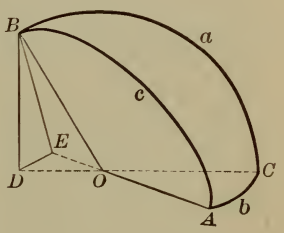


FIG. 88

The auxiliary plane BDE will now cut both CO and AO produced beyond the centre O; and we have

$$\cos c = -OE = -OD \cos DOE$$
$$= -(-\cos a)\cos b = \cos a \cos b.$$

Likewise, the other formulas, [39]–[43], hold true in this case.

Again, suppose that both the legs a and b are greater than 90°. In this case the plane BDE will cut CO produced beyond O, and AO between A and O; and we have

$$\cos c = OE = OD \cos DOE$$
$$= (-\cos a)(-\cos b) = \cos a \cos b,$$

a result agreeing with [38].

Likewise the other formulas, [39]–[43], hold true in this case.

Like results may be obtained in all cases.

In other words, Formulas [38]–[43] are universally true.

EXERCISE XXXII

1. Show, by aid of Formula [38], p. 144, that the hypotenuse of a right spherical triangle is *less than* or *greater than* 90°, according as the two legs are *alike* or *unlike* in kind.

2. Show, by aid of Formula [41], that in a right spherical triangle each leg and the opposite angle are always alike in kind.

3. What inferences may be drawn from Formulas [38]–[43] respecting the values of the other parts: (i) if $c = 90°$; (ii) if $a = 90°$; (iii) if $c = 90°$ and $a = 90°$; (iv) if $a = 90°$ and $b = 90°$?

Deduce from Formulas [38]–[43] and Formulas [18]–[23] the following formulas:

4. $\tan^2 \frac{1}{2} b = \tan \frac{1}{2}(c - a) \tan \frac{1}{2}(c + a)$.

HINT. Substitute in Formula [18] the value of cos b from [38].

5. $\tan^2(45° - \frac{1}{2} A) = \tan \frac{1}{2}(c - a) \cot \frac{1}{2}(c + a)$.

6. $\tan^2 \frac{1}{2} B = \sin(c - a) \csc(c + a)$.

7. $\tan^2 \frac{1}{2} c = -\cos(A + B) \sec(A - B)$.

8. $\tan^2 \frac{1}{2} a = \tan[\frac{1}{2}(A + B) - 45°] \tan[\frac{1}{2}(A - B) + 45°]$.

9. $\tan^2(45° - \frac{1}{2} c) = \tan \frac{1}{2}(A - a) \cot \frac{1}{2}(A + a)$.

10. $\tan^2(45° - \frac{1}{2} b) = \sin(A - a) \csc(A + a)$.

11. $\tan^2(45° - \frac{1}{2} B) = \tan \frac{1}{2}(A - a) \tan \frac{1}{2}(A + a)$.

SECTION LI

NAPIER'S RULES

The ten formulas deduced in Sect. L express the relations of five parts of a right triangle, the three sides and the two oblique angles. All these relations may be shown to

follow from two very useful rules, devised by Baron Napier, the inventor of logarithms.

For this purpose the right angle (not entering the formulas) is not taken into account, and instead of the hypotenuse and the two oblique angles their respective *complements* are employed; so that the five parts considered by Napier's Rules are: *a*, *b*, *Co. A*, *Co. c*, *Co. B*.

Any one of these parts may be called a *middle* part; and then the two parts immediately adjacent are called *adjacent* parts, and the other two are called *opposite* parts. Napier's Rules are

RULE I. *The sine of any middle part is equal to the product of the* **tan***gents of the* **ad***jacent parts.*

RULE II. *The sine of any middle part is equal to the product of the* **cos***ines of the* **op***posite parts.*

These rules are easily remembered by the expressions **tan. ad.** and **cos. op.**

The correctness of these rules may be easily shown by taking in turn each of the five parts as middle part, and comparing the resulting equations with the equations contained in Formulas [38]–[43], p. 144.

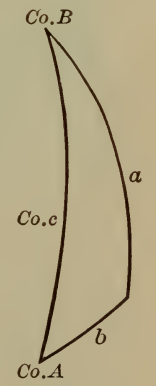


FIG. 89

For example, let *Co. c* be taken as middle part; then *Co. A* and *Co. B* are the adjacent parts, and *a* and *b* the opposite parts, as is very plainly seen in Fig. 89. Then, by Napier's Rules,

$$\sin (Co.\ c) = \tan (Co.\ A) \tan (Co.\ B),$$

or $$\cos c = \cot A \cot B;$$

$$\sin (Co.\ c) = \cos a \cos b,$$

or $$\cos c = \cos a \cos b.$$

These results agree respectively with Formulas [43] and [38], p. 144.

EXERCISE XXXIII

1. Show that Napier's Rules lead to the equations contained in Formulas [39], [40], [41], and [42].

2. What will Napier's Rules become if we take as the five parts of the triangle the hypotenuse, the two oblique angles, and the *complements* of the two legs?

SECTION LII

SOLUTION OF THE RIGHT SPHERICAL TRIANGLE

By means of Formulas [38]–[43], p. 144, we can solve a right triangle in all possible cases. In every case two parts besides the right angle must be given.

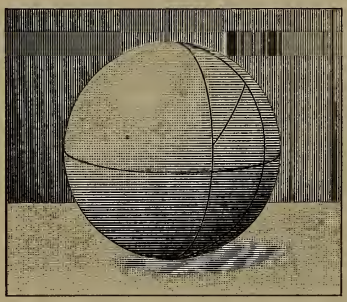


FIG. 90

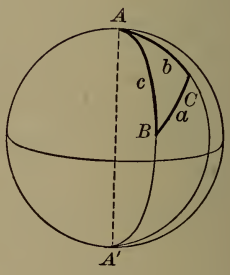


FIG. 91

CASE I

Given the two legs a *and* b.

From Formulas [38] and [42], p. 144, we obtain

$$\cos c = \cos a \cos b,$$
$$\tan A = \tan a \csc b,$$
$$\tan B = \tan b \csc a.$$

For a check use $\cos c = \cot A \cot B$, [43], p. 144.

EXAMPLE. Given $a = 27° 28' 36''$, $b = 51° 12' 8''$; solve the triangle.

$$\log \cos a = 9.94802$$
$$\log \cos b = 9.79697$$
$$\log \cos c = \overline{9.74499}$$
$$c = 56° 13' 41''$$

$$\log \tan a = 9.71605$$
$$\log \csc b = \overline{0.10826}$$
$$\log \tan A = \overline{9.82431}$$
$$A = 33° 42' 51''$$

$$\log \tan b = 10.09476$$
$$\log \csc a = \overline{0.33594}$$
$$\log \tan B = \overline{10.43070}$$
$$B = 69° 38' 54''$$

Check.

$$\log \cot A = 10.17569$$
$$\log \cot B = \overline{9.56930}$$
$$\log \cos c = \overline{9.74499}$$

CASE II

Given the hypotenuse c *and the leg* a.

From Formulas [38], [39], and [40], p. 144, we obtain

$$\cos b = \cos c \sec a,$$
$$\sin A = \sin a \csc c,$$
$$\cos B = \tan a \cot c.$$

For a check use $\cos B = \cos b \sin A$, [41], p. 144.

Although two angles in general correspond to $\sin A$, one acute, the other obtuse, yet in this case it is easy to determine whether A is acute or obtuse since A and a must be alike in kind. (See Example 2, Exercise XXXII, p. 146.)

CASE III

Given the leg a *and the opposite angle* A.

From Formulas [39], [42], and [41], we obtain

$$\sin c = \sin a \csc A,$$
$$\sin b = \tan a \cot A,$$
$$\sin B = \sec a \cos A.$$

Or, from [38] and [40], p. 144, we obtain

$$\cos b = \cos c \sec a,$$
$$\cos B = \tan a \cot c.$$

For a check use $\sin b = \sin c \sin B$, [39], p. 144.

When c has been computed, b and B are determined by these values of their cosines; but, since c must be found from its sine, c may have, in general, two values which are supplements of each other. This case, therefore, really admits of two solutions.

In fact, if the sides b and c are extended until they meet in A' (Fig. 91), the two right triangles ABC and $A'BC$ have the side a in common, and the angle $A = A'$. Also, $A'C = 180° - b$, $A'B = 180° - c$, and $\angle A'BC = 180° - B$. Hence, if ABC is one solution, $A'BC$ is the other.

Case IV

Given the leg a *and the adjacent angle* B.

From Formulas [40], [42], and [41], p. 144, we obtain

$$\tan c = \tan a \sec B,$$
$$\tan b = \sin a \tan B,$$
$$\cos A = \cos a \sin B.$$

For a check use $\cos A = \tan b \cot c$, [40], p. 144.

Case V

Given the hypotenuse c *and the angle* A.

From Formulas [39], [40], and [43], p. 144, we obtain

$$\sin a = \sin c \sin A,$$
$$\tan b = \tan c \cos A,$$
$$\cot B = \cos c \tan A.$$

Here a is determined by $\sin a$, since a and A must be alike in kind. (See Example 2, Exercise XXXII, p. 146.)

For a check use $\sin a = \tan b \cot B$, [42], p. 144.

Case VI

Given the two angles A *and* B.

From Formulas [43] and [41], p. 144, we obtain

$$\cos c = \cot A \cot B,$$
$$\cos a = \cos A \csc B,$$
$$\cos b = \cos B \csc A.$$

For a check use $\cos c = \cos a \cos b$, [38], p. 144.

Note 1. In Case I (a and b given), if c is very near 0° or 180°, it may be found with greater accuracy by first computing B, and then computing c, as in Case IV.

Note 2. In Case II (c and a given), if b is very near 0° or 180°, it may be computed more accurately by means of the derived formula

$$\tan^2 \tfrac{1}{2} b = \tan \tfrac{1}{2}(c - a) \tan \tfrac{1}{2}(c + a). \qquad \text{(Ex. 4, Sect. L)}$$

If A is so near 90° that it cannot be found accurately in the tables, it may be computed from the derived formula

$$\tan^2(45° - \tfrac{1}{2} A) = \tan \tfrac{1}{2}(c - a) \cot \tfrac{1}{2}(c + a). \qquad \text{(Ex. 5, Sect. L)}$$

If B cannot be found accurately, we may use the formula

$$\tan^2 \tfrac{1}{2} B = \sin(c - a) \csc(c + a). \qquad \text{(Ex. 6, Sect. L)}$$

Note 3. In Case III (a and A given), when the formulas do not give accurate results, we may employ the derived formulas

$$\tan^2(45° - \tfrac{1}{2} c) = \tan \tfrac{1}{2}(A - a) \cot \tfrac{1}{2}(A + a), \qquad \text{(Ex. 9, Sect. L)}$$
$$\tan^2(45° - \tfrac{1}{2} b) = \sin(A - a) \csc(A + a), \qquad \text{(Ex. 10, Sect. L)}$$
$$\tan^2(45° - \tfrac{1}{2} B) = \tan \tfrac{1}{2}(A - a) \tan \tfrac{1}{2}(A + a) \qquad \text{(Ex. 11, Sect. L)}$$

Note 4. In Case IV (a and B given), if A is near 0° or 180°, it may be more accurately found by first computing b and then finding A.

Note 5. In Case V (c and A given), if a is near 90°, it may be found by first computing b, and then computing a by Formula [42], p. 144.

Note 6. In Case VI (A and B given), for unfavorable values of the sides greater accuracy may be obtained by means of the derived formulas

$$\tan^2 \tfrac{1}{2} c = -\cos(A + B) \sec(A - B), \qquad \text{(Ex. 7, Sect. L)}$$
$$\tan^2 \tfrac{1}{2} a = \tan[\tfrac{1}{2}(A + B) - 45°] \tan[45° + \tfrac{1}{2}(A - B)], \qquad \text{(Ex. 8, Sect. L)}$$
$$\tan^2 \tfrac{1}{2} b = \tan[\tfrac{1}{2}(A + B) - 45°] \tan[45° - \tfrac{1}{2}(A - B)].$$

NOTE 7. In Cases I, IV, and V, the solution is always possible. In the other Cases, in order that the solution may be possible, it is necessary and sufficient that in Case II $\sin a < \sin c$; in Case III, that a and A be alike in kind, and $\sin A > \sin a$; in Case VI, that $A + B + C > 180°$, and the difference between A and $B < 90°$.

NOTE 8. It is easy to trace analogies between the formulas for solving right spherical triangles and those for solving right plane triangles. The former become identical with the latter if we suppose the radius of the sphere to be infinite in length. Then the cosines of the sides become each equal to 1, and the ratios of the *sines* of the sides and of the *tangents* of the sides must be taken as equal to the ratios of the sides themselves.

NOTE 9. In solving spherical triangles, the algebraic sign of the functions must receive careful attention. Write the sign of each function just above the function. Then the sign of the function in the first member of the equation is + or − according as the law of signs makes the second member of the equation positive or negative. (See Example 1, p. 175.)

If the function is a cosine, tangent, or cotangent, the + sign shows the angle $< 90°$; the − sign shows the angle $> 90°$, and the *supplement* of the angle obtained from the table must be taken.

If the function is a sine, since the sine of an angle and of its supplement are the same, the acute angle obtained from the table and its supplement must be considered as solutions, unless there are other conditions that remove the ambiguity. For conditions that remove the ambiguity, see Examples 1 and 2 in Exercise XXXII, p. 146.

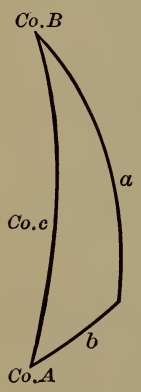


FIG. 92

It is always easy to find the required formula by means of Napier's Rules. In applying these rules we must choose for the middle part that one of the three parts which will make the two given parts either adjacent parts or opposite parts.

Example: *Given* a *and* B; *solve the triangle.*

To find b, take a as the middle part; then b and *Co.* B are the adjacent parts; and, by Rule I,

$$\sin a = \tan b \cot B.$$

Whence, $$\tan b = \sin a \tan B.$$

To find c, take $Co.\,B$ as the middle part; then a and $Co.\,c$ are the adjacent parts; and, by Rule I,

$$\cos B = \tan a \cot c.$$

Whence, $$\tan c = \tan a \sec B.$$

To find A, take $Co.\,A$ as the middle part; then a and $Co.\,B$ are the opposite parts; and, by Rule II,

$$\cos A = \cos a \sin B.$$

EXERCISE XXXIV

Solve the following right triangles, taking for the given parts in each case those printed in columns I and II:

	I	II	III	IV	V
	a	b	c	A	B
1	36° 27′	43° 32′ 31″	54° 20′	46° 59′ 43″	57° 59′ 19″
2	86° 40′	32° 40′	87° 11′ 40″	88° 11′ 58″	32° 42′ 39″
3	50°	36° 54′ 49″	59° 4′ 26″	63° 15′ 13″	44° 26′ 22″
4	120° 10′	150° 59′ 44″	63° 55′ 43″	105° 44′ 21″	147° 19′ 47″
	c	a	b	A	B
5	55° 9′ 32″	22° 15′ 7″	51° 53′	27° 28′ 38″	73° 27′ 11″
6	23° 49′ 51″	14° 16′ 35″	19° 17′	37° 36′ 49″	54° 49′ 23″
7	44° 33′ 17″	32° 9′ 17″	32° 41′	49° 20′ 16″	50° 19′ 16″
8	97° 13′ 4″	132° 14′ 12″	79° 13′ 38″	131° 43′ 50″	81° 58′ 53″
	a	A	c	b	B
9	77° 21′ 50″	83° 56′ 40″	78° 53′ 20″ 101° 6′ 40″	28° 14′ 31″ 151° 45′ 29″	28° 49′ 57″ 151° 10′ 3″
10	77° 21′ 50″	40° 40′ 40″	impossible		

NOTE. The values in the last three columns of Example 9 cannot be combined promiscuously with those given in columns I and II.

If $a < 90°$, with the value of $b > 90°$ must be taken $B > 90°$ and $c > 90°$; while with the value of $b < 90°$ must be taken, for the same reason, $B < 90°$ and $c < 90°$. (See Examples 1 and 2, Exercise XXXII, p. 146.)

	I	II	III	IV	V
	a	B	c	b	A
11	92° 47′ 32″	50° 2′ 1″	91° 47′ 40″	49° 59′ 58″	92° 8′ 23″
12	2° 0′ 55″	12° 40′	2° 3′ 56″	0° 27′ 10″	77° 20′ 28″
13	20° 20′ 20″	38° 10′ 10″	25° 14′ 38″	15° 16′ 50″	54° 35′ 17″
14	54° 30′	35° 30′	59° 51′ 21″	30° 8′ 39″	70° 17′ 35″
	c	A	a	b	B
15	69° 25′ 11″	54° 54′ 42″	50°	56° 50′ 49″	63° 25′ 4″
16	112° 48′	56° 11′ 56″	50°	127° 4′ 30″	120° 3′ 50″
17	46° 40′ 12″	37° 46′ 9″	26° 27′ 24″	39° 57′ 42″	62° 0′ 4″
18	118° 40′ 1″	128° 0′ 4″	136° 15′ 32″	48° 23′ 38″	58° 27′ 4″
	A	B	a	b	c
19	63° 15′ 12″	135° 33′ 39″	50° 0′ 4″	143° 5′ 12″	120° 55′ 34″
20	116° 43′ 12″	116° 31′ 25″	120° 10′ 3″	119° 59′ 46″	75° 26′ 58″
21	46° 59′ 42″	57° 59′ 17″	36° 27′	43° 32′ 30″	54° 20′
22	90°	88° 24′ 35″	90°	88° 24′ 35″	90°

23. Define a quadrantal triangle, and show how its solution may be reduced to that of the right triangle.

24. Solve the quadrantal triangle the sides of which are

$$a = 174° 12′ 49″,\ b = 94° 8′ 20″,\ c = 90°.$$

25. Solve the quadrantal triangle in which

$$c = 90°,\ A = 110° 47′ 50″,\ B = 135° 35′ 34.5″.$$

26. Given in a spherical triangle A, C, and c each equal to 90°; solve the triangle.

27. Given $A = 60°$, $C = 90°$, and $c = 90°$; solve the triangle.

28. In a right spherical triangle, given $A = 42° 24′ 9″$, $B = 9° 4′ 11″$; solve the triangle.

29. In a right spherical triangle, given $a = 119° 11′$, $B = 126° 54′$; solve the triangle.

30. In a right spherical triangle, given $c = 50°$, $b = 44°18'39''$; solve the triangle.

31. In a right spherical triangle, given $A = 156°\ 20'\ 30''$, $a = 65°\ 15'\ 45''$; solve the triangle.

32. In a right spherical triangle, given $A = 74°\ 12'\ 31''$, $c = 64°\ 28'\ 47''$; solve the triangle.

33. In a right spherical triangle, given $a = 112°\ 42'\ 38''$, $B = 44°\ 28'\ 44''$; solve the triangle.

34. In a right spherical triangle, given $b = 48°\ 12'\ 48''$, $A = 108°\ 14'\ 14''$; solve the triangle.

35. In a right spherical triangle, given $A = 122°\ 58'\ 47''$, $B = 104°\ 17'\ 55''$; solve the triangle.

36. If the legs a and b of a right spherical triangle are equal, show that $\cos a = \cot A = \sqrt{\cos c}$.

37. In a right spherical triangle show that

$$\cos^2 A \sin^2 c = \sin(c + a)\sin(c - a).$$

38. In a right spherical triangle show that

$$\tan a \cos c = \sin b \cot B.$$

39. In a right spherical triangle show that

$$\sin^2 A = \cos^2 B + \sin^2 a \sin^2 B.$$

40. In a right spherical triangle show that

$$\sin(b + c) = 2\cos^2 \tfrac{1}{2} A \cos b \sin c.$$

41. In a right spherical triangle show that

$$\sin(c - b) = 2\sin^2 \tfrac{1}{2} A \cos b \sin c.$$

42. If in a right spherical triangle, p denotes the arc of the great circle passing through the vertex of the right angle and perpendicular to the hypotenuse, m and n the segments of the hypotenuse made by this arc adjacent to the legs a and b, show that (i) $\tan^2 a = \tan c \tan m$, (ii) $\sin^2 p = \tan m \tan n$.

SECTION LIII

SOLUTION OF THE ISOSCELES SPHERICAL TRIANGLE

An arc of a great circle, passed through the vertex of an isosceles spherical triangle and the middle point of the base (Fig. 94), divides the triangle into two equivalent right spherical triangles. Hence, the solution of an isosceles spherical triangle may be reduced to that of a right spherical triangle.

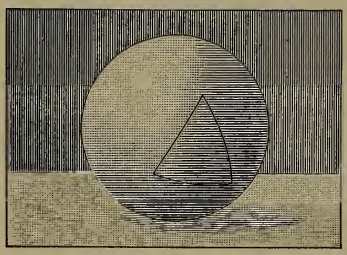


FIG. 93

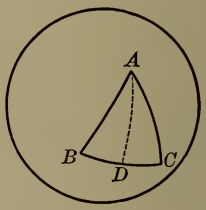


FIG. 94

Likewise the solution of a regular spherical polygon may be reduced to that of a right spherical triangle. Arcs of great

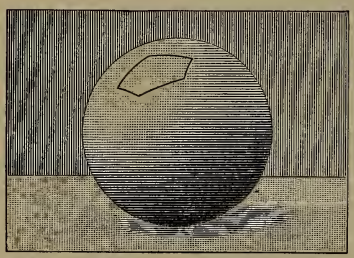


FIG. 95

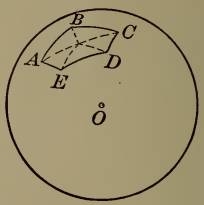


FIG. 96

circles, passed through the centre of the polygon and the vertices (Fig. 96), divide the polygon into equal isosceles triangles;

and each one of these equal isosceles triangles may be divided into two equivalent right triangles.

A regular spherical polygon (Fig. 97) is the polygon formed by the intersections of the spherical surface with the faces of a regular pyramid whose vertex is at the centre of the sphere.

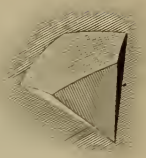


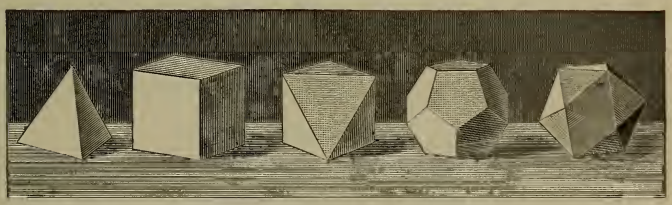
FIG. 97

EXERCISE XXXV

1. In an isosceles spherical triangle, given the base b and the side a; find A the angle at the base, B the angle at the vertex, and h the altitude.

2. In an equilateral spherical triangle, given the side a; find the angle A.

3. Given the side a of a regular spherical polygon of n sides; find the angle A of the polygon, the distance R from the centre of the polygon to one of the vertices, and the distance r from the centre to the middle point of one of the sides.



Tetrahedron Hexahedron Octahedron Dodecahedron Icosahedron

FIG. 98

4. Compute the dihedral angles made by the faces of the five regular polyhedrons (Fig. 98).

5. A spherical square is a regular spherical quadrilateral. Find the angle A of the square, having given the side a.

CHAPTER VIII

THE OBLIQUE SPHERICAL TRIANGLE

SECTION LIV

FUNDAMENTAL FORMULAS

Let ABC (Fig. 99) be an oblique spherical triangle, a, b, c its three sides, A, B, C the angles opposite a, b, c, respectively.

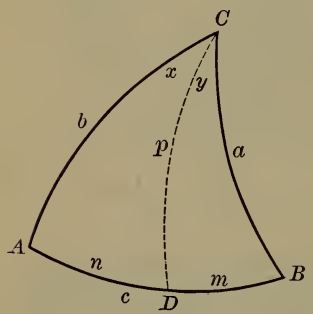


FIG. 99

Through C draw CD, an arc of a great circle, perpendicular to the side AB, meeting AB at D. For brevity let

$$CD = p,\ AD = n,\ BD = m,$$
$$\angle ACD = x,\ \angle BCD = y.$$

1. By [39], p. 144, in the right triangles BDC and ADC,

$$\sin p = \sin a \sin B,$$
and $$\sin p = \sin b \sin A.$$

Therefore, $\mathbf{\sin a \sin B = \sin b \sin A}$
similarly, $\mathbf{\sin a \sin C = \sin c \sin A}$
and $\mathbf{\sin b \sin C = \sin c \sin B}$. [44]

These equations may also be written in the form of proportions:

$$\sin a : \sin b : \sin c = \sin A : \sin B : \sin C.$$

That is, *the sines of the sides of a spherical triangle are proportional to the sines of the opposite angles.*

In Fig. 99 the arc CD cuts the side AB within the triangle. If CD falls without the triangle, for instance to the right of CB, $\sin(180° - B)$ would then be employed instead of $\sin B$. But $\sin(180° - B) = \sin B$ (Sect. XXV, p. 48). Hence, the Formulas [44] hold true in all cases.

2. In the right triangle BDC, by [38], p. 144,

$$\cos a = \cos p \cos m$$
$$= \cos p \cos(c - n)$$

by [9], p. 56,
$$= \cos p \cos c \cos n + \cos p \sin c \sin n.$$

Now, in the right triangle ADC, by [38], p. 144,

$$\cos p \cos n = \cos b.$$

Whence,
$$\cos p = \cos b \sec n,$$

and
$$\cos p \sin n = \cos b \tan n.$$

By [40], p. 144, $\tan n = \tan b \cos A$.

$$\therefore \cos p \sin n = \cos b \tan b \cos A$$
$$= \sin b \cos A.$$

Substituting these values of $\cos p \cos n$ and $\cos p \sin n$ in the value of $\cos a$, we obtain

and similarly,
$$\left.\begin{aligned} \cos a &= \cos b \cos c + \sin b \sin c \cos A \\ \cos b &= \cos a \cos c + \sin a \sin c \cos B \\ \cos c &= \cos a \cos b + \sin a \sin b \cos C \end{aligned}\right\}. \quad [45]$$

3. In the right triangle ADC, by [41], p. 144,

$$\cos A = \cos p \sin x$$
$$= \cos p \sin(C - y)$$

by [8], p. 56,
$$= \cos p \sin C \cos y - \cos p \cos C \sin y.$$

Now, by [41], p. 144,

$$\cos p \sin y = \cos B.$$

Therefore,
$$\cos p = \cos B \csc y,$$

and
$$\cos p \cos y = \cos B \cot y$$

by [43], p. 144,
$$= \cos B \tan B \cos a$$
$$= \sin B \cos a.$$

Substituting these values of $\cos p \sin y$ and $\cos p \cos y$ in the value of $\cos A$, we obtain

$$\left.\begin{array}{l} \cos A = -\cos B \cos C + \sin B \sin C \cos a \\ \text{and similarly,} \quad \cos B = -\cos A \cos C + \sin A \sin C \cos b \\ \cos C = -\cos A \cos B + \sin A \sin B \cos c \end{array}\right\}. \quad [46]$$

Formulas [45] and [46] are also universally true; for the same equations are obtained when the arc CD cuts the side AB without the triangle.

EXERCISE XXXVI

1. What do Formulas [44] become if $A = 90°$? if $B = 90°$? if $C = 90°$? if $a = 90°$? if $A = B = 90°$? if $a = b = 90°$?

2. What do Formulas [45] become if $A = 90°$? if $B = 90°$? if $C = 90°$? if $A = B = C = 90$?

3. What does the first of Formulas [45] become if $A = 0°$? if $A = 90°$? if $A = 180°$?

4. From Formulas [45] deduce Formulas [46], by means of the relations between polar triangles (Theorem 4, p. 141).

SECTION LV

FORMULAS FOR THE HALF ANGLES AND SIDES

From the first equation of [45], p. 159,

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

Therefore, $$1 - \cos A = \frac{\sin b \sin c + \cos b \cos c - \cos a}{\sin b \sin c}$$

by [9], p. 56, $$= \frac{\cos (b - c) - \cos a}{\sin b \sin c}$$

by [23], p. 59, $$= \frac{-2 \sin \tfrac{1}{2}(a + b - c) \sin \tfrac{1}{2}(b - c - a)}{\sin b \sin c}.$$

Also, $1 + \cos A = \dfrac{\sin b \sin c - \cos b \cos c + \cos a}{\sin b \sin c}$

by [5], p. 54, $= \dfrac{\cos a - \cos(b + c)}{\sin b \sin c}$

by [23], p. 59, $= \dfrac{-2 \sin \frac{1}{2}(a+b+c) \sin \frac{1}{2}(a-b-c)}{\sin b \sin c}.$

Since by [16], p. 58,

$$1 - \cos A = 2 \sin^2 \tfrac{1}{2} A,$$

$$\sin^2 \tfrac{1}{2} A = \sin \tfrac{1}{2}(a + b - c) \sin \tfrac{1}{2}(a - b + c) \csc b \csc c;$$

and since by [17], p. 58,

$$1 + \cos A = 2 \cos^2 \tfrac{1}{2} A,$$

$$\cos^2 \tfrac{1}{2} A = \sin \tfrac{1}{2}(a + b + c) \sin \tfrac{1}{2}(b + c - a) \csc b \csc c.$$

Now let $\tfrac{1}{2}(a + b + c) = s.$

Then, $\tfrac{1}{2}(b + c - a) = s - a,$

$\tfrac{1}{2}(a - b + c) = s - b,$

and $\tfrac{1}{2}(a + b - c) = s - c.$

Then, by substitution and extraction of the square root,

$$\left.\begin{aligned} \sin \tfrac{1}{2} \mathrm{A} &= \sqrt{\sin(\mathrm{s} - \mathrm{b}) \sin(\mathrm{s} - \mathrm{c}) \csc \mathrm{b} \csc \mathrm{c}} \\ \cos \tfrac{1}{2} \mathrm{A} &= \sqrt{\sin \mathrm{s} \sin(\mathrm{s} - \mathrm{a}) \csc \mathrm{b} \csc \mathrm{c}} \\ \tan \tfrac{1}{2} \mathrm{A} &= \sqrt{\csc \mathrm{s} \csc(\mathrm{s} - \mathrm{a}) \sin(\mathrm{s} - \mathrm{b}) \sin(\mathrm{s} - \mathrm{c})} \end{aligned}\right\}. \quad [47]$$

In like manner, it may be shown that

$$\left.\begin{aligned} \sin \tfrac{1}{2} \mathrm{B} &= \sqrt{\sin(\mathrm{s} - \mathrm{a}) \sin(\mathrm{s} - \mathrm{c}) \csc \mathrm{a} \csc \mathrm{c}} \\ \cos \tfrac{1}{2} \mathrm{B} &= \sqrt{\sin \mathrm{s} \sin(\mathrm{s} - \mathrm{b}) \csc \mathrm{a} \csc \mathrm{c}} \\ \tan \tfrac{1}{2} \mathrm{B} &= \sqrt{\csc \mathrm{s} \csc(\mathrm{s} - \mathrm{b}) \sin(\mathrm{s} - \mathrm{a}) \sin(\mathrm{s} - \mathrm{c})} \\ \sin \tfrac{1}{2} \mathrm{C} &= \sqrt{\sin(\mathrm{s} - \mathrm{a}) \sin(\mathrm{s} - \mathrm{b}) \csc \mathrm{a} \csc \mathrm{b}} \\ \cos \tfrac{1}{2} \mathrm{C} &= \sqrt{\sin \mathrm{s} \sin(\mathrm{s} - \mathrm{c}) \csc \mathrm{a} \csc \mathrm{b}} \\ \tan \tfrac{1}{2} \mathrm{C} &= \sqrt{\csc \mathrm{s} \csc(\mathrm{s} - \mathrm{c}) \sin(\mathrm{s} - \mathrm{a}) \sin(\mathrm{s} - \mathrm{b})} \end{aligned}\right\}. \quad [47]$$

Again, from the first equation of [46], p. 160,

$$\cos a = \frac{\cos B \cos C + \cos A}{\sin B \sin C}.$$

Therefore,

$$1 - \cos a = \frac{\sin B \sin C - \cos B \cos C - \cos A}{\sin B \sin C}$$

by [5], p. 54, $$= \frac{-\cos (B + C) - \cos A}{\sin B \sin C}$$

by [22], p. 59, $$= \frac{-2 \cos \tfrac{1}{2}(B + C + A) \cos \tfrac{1}{2}(B + C - A)}{\sin B \sin C};$$

and $$1 + \cos a = \frac{\sin B \sin C + \cos B \cos C + \cos A}{\sin B \sin C}$$

by [9], p. 56, $$= \frac{\cos (B - C) + \cos A}{\sin B \sin C}$$

by [22], p. 59, $$= \frac{2 \cos \tfrac{1}{2}(B - C + A) \cos \tfrac{1}{2}(B - C - A)}{\sin B \sin C}.$$

Since by [16], p. 58,

$$1 - \cos a = 2 \sin^2 \tfrac{1}{2} a,$$

$$\sin^2 \tfrac{1}{2} a = -\cos \tfrac{1}{2}(B + C + A) \cos \tfrac{1}{2}(B + C - A) \csc B \csc C;$$

and since by [17], p. 58,

$$1 + \cos a = 2 \cos^2 \tfrac{1}{2} a,$$

$$\cos^2 \tfrac{1}{2} a = \cos \tfrac{1}{2}(B - C + A) \cos \tfrac{1}{2}(B - C - A) \csc B \csc C.$$

Now let $\tfrac{1}{2}(A + B + C) = S.$

Then, $\tfrac{1}{2}(B + C - A) = S - A,$

$\tfrac{1}{2}(A - B + C) = S - B,$

and $\tfrac{1}{2}(A + B - C) = S - C.$

Then, by substitution and extraction of the square root,

$$\left.\begin{aligned} \sin \tfrac{1}{2} a &= \sqrt{-\cos S \cos (S - A) \csc B \csc C} \\ \cos \tfrac{1}{2} a &= \sqrt{\cos (S - B) \cos (S - C) \csc B \csc C} \\ \tan \tfrac{1}{2} a &= \sqrt{-\cos S \cos (S - A) \sec (S - B) \sec (S - C)} \end{aligned}\right\}. \quad [48]$$

And, in like manner,

$$\left.\begin{aligned}
\sin\tfrac{1}{2}\mathrm{b} &= \sqrt{-\cos\mathrm{S}\cos(\mathrm{S}-\mathrm{B})\csc\mathrm{A}\csc\mathrm{C}} \\
\cos\tfrac{1}{2}\mathrm{b} &= \sqrt{\cos(\mathrm{S}-\mathrm{A})\cos(\mathrm{S}-\mathrm{C})\csc\mathrm{A}\csc\mathrm{C}} \\
\tan\tfrac{1}{2}\mathrm{b} &= \sqrt{-\cos\mathrm{S}\cos(\mathrm{S}-\mathrm{B})\sec(\mathrm{S}-\mathrm{A})\sec(\mathrm{S}-\mathrm{C})} \\
\sin\tfrac{1}{2}\mathrm{c} &= \sqrt{-\cos\mathrm{S}\cos(\mathrm{S}-\mathrm{C})\csc\mathrm{A}\csc\mathrm{B}} \\
\cos\tfrac{1}{2}\mathrm{c} &= \sqrt{\cos(\mathrm{S}-\mathrm{A})\cos(\mathrm{S}-\mathrm{B})\csc\mathrm{A}\csc\mathrm{B}} \\
\tan\tfrac{1}{2}\mathrm{c} &= \sqrt{-\cos\mathrm{S}\cos(\mathrm{S}-\mathrm{C})\sec(\mathrm{S}-\mathrm{A})\sec(\mathrm{S}-\mathrm{B})}
\end{aligned}\right\}. \quad [48]$$

SECTION LVI

GAUSS'S EQUATIONS AND NAPIER'S ANALOGIES

By [5], p. 54,

$$\cos\tfrac{1}{2}(A+B) = \cos\tfrac{1}{2}A\cos\tfrac{1}{2}B - \sin\tfrac{1}{2}A\sin\tfrac{1}{2}B;$$

or, by substituting for $\cos\frac{1}{2}A$, $\cos\frac{1}{2}B$, $\sin\frac{1}{2}A$, $\sin\frac{1}{2}B$, their values given in Formulas [47], p. 161, and reducing,

$$\cos\tfrac{1}{2}(A+B) = \sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}}\times\sqrt{\frac{\sin s\sin(s-b)}{\sin a\sin c}}$$

$$-\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}}\times\sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin a\sin c}}$$

$$= \frac{\sin s}{\sin c}\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a\sin b}} - \frac{\sin(s-c)}{\sin c}\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a\sin b}}$$

$$= \frac{\sin s - \sin(s-c)}{\sin c}\times\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a\sin b}}.$$

By [21], p. 59,

$$\sin s - \sin(s-c) = 2\cos\tfrac{1}{2}(s+s-c)\sin\tfrac{1}{2}(s-s+c)$$

$$= 2\cos(s-\tfrac{1}{2}c)\sin\tfrac{1}{2}c.$$

By [12], p. 58, $\sin c = 2\sin\frac{1}{2}c\cos\frac{1}{2}c$.

Again, by [47], p. 161,

$$\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a \sin b}} = \sin \tfrac{1}{2} C.$$

Substituting in the value of $\cos \frac{1}{2}(A+B)$, we have

$$\cos \tfrac{1}{2}(A+B) = \frac{2 \sin \frac{1}{2} c \cos(s - \frac{1}{2} c)}{2 \sin \frac{1}{2} c \cos \frac{1}{2} c} \sin \tfrac{1}{2} C$$

$$= \frac{\cos(s - \frac{1}{2} c)}{\cos \frac{1}{2} c} \sin \tfrac{1}{2} C.$$

$$\therefore \cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2} c = \cos(s - \tfrac{1}{2} c) \sin \tfrac{1}{2} C.$$

Since $$s - \tfrac{1}{2} c = \tfrac{1}{2}(a+b),$$

$$\cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2} c = \cos \tfrac{1}{2}(a+b) \sin \tfrac{1}{2} C.$$

By proceeding in like manner with the values of

$$\sin \tfrac{1}{2}(A+B),\ \cos \tfrac{1}{2}(A-B),\ \text{and}\ \sin \tfrac{1}{2}(A-B),$$

three analogous equations are obtained.

The four equations,

$$\left.\begin{aligned} \cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2} c &= \cos \tfrac{1}{2}(a+b) \sin \tfrac{1}{2} C \\ \sin \tfrac{1}{2}(A+B) \cos \tfrac{1}{2} c &= \cos \tfrac{1}{2}(a-b) \cos \tfrac{1}{2} C \\ \cos \tfrac{1}{2}(A-B) \sin \tfrac{1}{2} c &= \sin \tfrac{1}{2}(a+b) \sin \tfrac{1}{2} C \\ \sin \tfrac{1}{2}(A-B) \sin \tfrac{1}{2} c &= \sin \tfrac{1}{2}(a-b) \cos \tfrac{1}{2} C \end{aligned}\right\}, \qquad [49]$$

are called Gauss's Equations.

By dividing the second of Gauss's Equations by the first, the fourth by the third, the third by the first, and the fourth by the second, we obtain

$$\left.\begin{aligned} \tan \tfrac{1}{2}(A+B) &= \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \tfrac{1}{2} C \\ \tan \tfrac{1}{2}(A-B) &= \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \cot \tfrac{1}{2} C \\ \tan \tfrac{1}{2}(a+b) &= \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \tan \tfrac{1}{2} c \\ \tan \tfrac{1}{2}(a-b) &= \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \tan \tfrac{1}{2} c \end{aligned}\right\}. \qquad [50]$$

There will be other forms in each case, according as other elements of the triangle are used.

These equations are called Napier's Analogies.

In the first equation the factors $\cos \frac{1}{2}(a - b)$ and $\cot \frac{1}{2} C$ are always positive; therefore, $\tan \frac{1}{2}(A + B)$ and $\cos \frac{1}{2}(a + b)$ must always have like signs.

Hence, if $a + b < 180°$, $\cos \frac{1}{2}(a + b) > 0$, and $\tan \frac{1}{2}(A + B) > 0$. Hence, $A + B < 180°$.

If $a + b > 180°$, then $A + B > 180°$.

If $a + b = 180°$, $\cos \frac{1}{2}(a + b) = 0$, and $\tan \frac{1}{2}(A + B) = \infty$. Hence, $\frac{1}{2}(A + B) = 90°$, and $A + B = 180°$.

Conversely, it may be shown from the third equation, that $a + b$ is less than, greater than, or equal to 180°, according as $A + B$ is less than, greater than, or equal to 180°.

SECTION LVII

Case I

Given two sides a *and* b *and the included angle* C.

The angles A and B may be found by the first two of Napier's Analogies:

$$\tan \tfrac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \tfrac{1}{2} C;$$

$$\tan \tfrac{1}{2}(A - B) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cot \tfrac{1}{2} C.$$

After A and B have been found, the side c may be found by [44], p. 158, or by [50], p. 164; but it is better to use for this purpose Gauss's Equations, because they involve the functions of the same angles that occur in working Napier's Analogies. Any one of the equations may be used; for example,

$$\cos \tfrac{1}{2} c = \frac{\cos \tfrac{1}{2}(a+b)}{\cos \tfrac{1}{2}(A+B)} \sin \tfrac{1}{2} C.$$

EXAMPLE. Given $a = 73°\,58'\,54''$, $b = 38°\,45'$, $C = 46°\,33'\,41''$; solve the triangle.

$$\begin{aligned} a &= 73°\ 58'\ 54'', \\ b &= 38°\ 45'\ 0'', \\ C &= 46°\ 33'\ 41''. \end{aligned}$$

Hence,
$$\begin{aligned} \tfrac{1}{2}(a-b) &= 17°\ 36'\ 57'', \\ \tfrac{1}{2}(a+b) &= 56°\ 21'\ 57'', \\ \tfrac{1}{2} C &= 23°\ 16'\ 50.5''. \end{aligned}$$

$$\begin{aligned} \log \cos \tfrac{1}{2}(a-b) &= 9.97914 \\ \log \sec \tfrac{1}{2}(a+b) &= 0.25658 \\ \log \cot \tfrac{1}{2} C &= \underline{0.36626} \\ \log \tan \tfrac{1}{2}(A+B) &= \underline{0.60198} \\ \log \sec \tfrac{1}{2}(A+B) &= 0.61515 \\ \log \cos \tfrac{1}{2}(a+b) &= 9.74342 \\ \log \sin \tfrac{1}{2} C &= \underline{9.59685} \\ \log \cos \tfrac{1}{2} c &= \underline{9.95542} \\ \tfrac{1}{2} c &= 25°\ 31'12'' \end{aligned}$$

$$\begin{aligned} \log \sin \tfrac{1}{2}(a-b) &= 9.48092 \\ \log \csc \tfrac{1}{2}(a+b) &= 0.07956 \\ \log \cot \tfrac{1}{2} C &= \underline{0.36626} \\ \log \tan \tfrac{1}{2}(A-B) &= \underline{9.92674} \\ \tfrac{1}{2}(A+B) &= 75°\ 57'\ 40.8'' \\ \tfrac{1}{2}(A-B) &= 40°\ 11'\ 25.4'' \end{aligned}$$

$$\left\{ \begin{aligned} A &= 116°\ 9'\ 6'' \\ B &= 35°\ 46'\ 15'' \\ c &= 51°\ 2'\ 24'' \end{aligned} \right.$$

To test the accuracy of the work we may use the Rule of Sine Proportion given in Sect. LIV, p. 158.

If c only is desired, it may be found from [45], p. 159, without previously computing A and B. But the Formulas [45] are not adapted to logarithmic work. Instead of changing them to forms suitable for logarithms, we may use the following method, which leads to the same results and has the advantage that, in applying it, nothing has to be remembered except Napier's Rules:

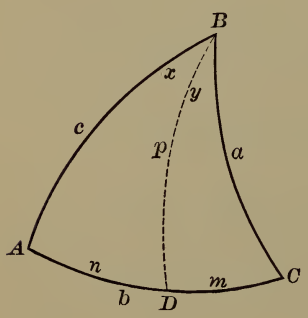


FIG. 100

Through B (Fig. 100) draw an arc of a great circle perpendicular to AC, cutting AC at D. Let $BD = p$, $CD = m$, $AD = n$.

By Rule I,

$$\cos C = \tan m \cot a.$$

Whence, $\tan m = \tan a \cos C.$

By Rule II,

$$\cos a = \cos m \cos p\,;\ \text{whence, } \cos p = \cos a \sec m.$$

$$\cos c = \cos n \cos p\,;\ \text{whence, } \cos p = \cos c \sec n.$$

Therefore, $\cos c \sec n = \cos a \sec m.$

Since $n = b - m,$

$$\cos c = \cos a \sec m \cos (b - m).$$

Now c may be computed from the two equations

$$\tan m = \tan a \cos C\,;$$

$$\cos c = \cos a \sec m \cos (b - m).$$

NOTE. If BD falls without the triangle, for instance to the right of BC, then $n = b + m$.

$$\therefore \cos c = \cos a \sec m \cos (b + m).$$

EXAMPLE. Given $a = 97° 30'$, $b = 55° 12'$, $C = 39° 58'$; find c.

$\log \tan a = 0.88057\ (n)$	$\log \cos a = 9.11570\ (n)$
$\log \cos C = 9.88447$	$\log \sec m = 0.77135\ (n)$
$\log \tan m = 0.76504\ (n)$	$\log \cos (b - m) = 9.85289$
$m = 99° 44' 49''$	$\log \cos c = 9.73994$
$b - m = -44° 32' 49''$	$c = 56° 40' 9''$

EXERCISE XXXVII

1. What are the formulas for computing a when b, c, and A are given; and for computing b when a, c, and B are given?

2. Given $a = 88° 12' 20''$, $b = 124° 7' 17''$, $C = 50° 2' 1''$;
find $A = 63° 15' 11''$, $B = 132° 17' 58''$, $c = 59° 4' 17''$.

3. Given $a = 120° 55' 35''$, $b = 88° 12' 20''$, $C = 47° 42' 1''$;
find $A = 129° 58' 2''$, $B = 63° 15' 8''$, $c = 55° 52' 40''$.

4. Given $b = 63° 15' 12''$, $c = 47° 42' 1''$, $A = 59° 4' 25''$;
find $B = 88° 12' 24''$, $C = 55° 52' 42''$, $a = 50° 1' 40''$.

5. Given $b = 69° 25' 11''$, $c = 109° 46' 19''$, $A = 54° 54' 42''$;
find $B = 56° 11' 57''$, $C = 123° 21' 12''$, $a = 67° 11' 47''$.

SECTION LVIII

Case II

Given two angles A *and* B, *and the included side* c.

The sides a and b may be found by the third and fourth of Napier's Analogies:

$$\tan \tfrac{1}{2}(a+b) = \frac{\cos \tfrac{1}{2}(A-B)}{\cos \tfrac{1}{2}(A+B)} \tan \tfrac{1}{2} c;$$

$$\tan \tfrac{1}{2}(a-b) = \frac{\sin \tfrac{1}{2}(A-B)}{\sin \tfrac{1}{2}(A+B)} \tan \tfrac{1}{2} c.$$

The angle C may be found by [44], p. 158, by Napier's second Analogy, [50], p. 164, or by one of Gauss's Equations, [49], p. 164. Thus, Gauss's second equation gives

$$\cos \tfrac{1}{2} C = \frac{\sin \tfrac{1}{2}(A+B)}{\cos \tfrac{1}{2}(a-b)} \cos \tfrac{1}{2} c.$$

Example. Given $A = 107°47'7''$, $B = 38°58'27''$, $c = 51°41'14''$; solve the triangle.

$$\begin{aligned} A &= 107°\ 47'\ \ 7'', \\ B &= \ 38°\ 58'\ 27'', \\ c &= \ 51°\ 41'\ 14''. \end{aligned}$$

Hence,
$$\begin{aligned} \tfrac{1}{2}(A-B) &= 34°\ 24'\ 20'', \\ \tfrac{1}{2}(A+B) &= 73°\ 22'\ 47'', \\ \tfrac{1}{2} c &= 25°\ 50'\ 37''. \end{aligned}$$

$$\begin{aligned} \log \cos \tfrac{1}{2}(A-B) &= 9.91648 \\ \log \sec \tfrac{1}{2}(A+B) &= 0.54359 \\ \log \tan \tfrac{1}{2} c &= 9.68517 \\ \log \tan \tfrac{1}{2}(a+b) &= 0.14524 \\ \log \sin \tfrac{1}{2}(A+B) &= 9.98146 \\ \log \sec \tfrac{1}{2}(a-b) &= 0.01703 \\ \log \cos \tfrac{1}{2} c &= 9.95423 \\ \log \cos \tfrac{1}{2} C &= 9.95272 \\ \tfrac{1}{2} C &= 26°\ 15'\ 10'' \end{aligned}$$

$$\begin{aligned} \log \sin \tfrac{1}{2}(A-B) &= 9.75208 \\ \log \csc \tfrac{1}{2}(A+B) &= 0.01854 \\ \log \tan \tfrac{1}{2} c &= 9.68517 \\ \log \tan \tfrac{1}{2}(a-b) &= 9.45579 \\ \tfrac{1}{2}(a+b) &= 54°\ 24'\ 24.4'' \\ \tfrac{1}{2}(a-b) &= 15°\ 56'\ 25.5'' \end{aligned}$$

$$\begin{cases} a = 70°\ 20'\ 50'' \\ b = 38°\ 27'\ 59'' \\ C = 52°\ 30'\ 20'' \end{cases}$$

If the angle C alone is wanted, we proceed as in Case I, p. 166, when the side c alone is desired.

Let (Fig. 101) $\angle ABD = x$, $\angle CBD = y$, $BD = p$; then,

Rule I, $\cos c = \cot x \cot A$.

Whence, $\cot x = \tan A \cos c$.

Rule II, $\cos A = \cos p \sin x$.

Whence, $\cos p = \cos A \csc x$.

Rule II, $\cos C = \cos p \sin y$.

Whence, $\cos p = \cos C \csc y$.

$\therefore \cos C = \cos A \csc x \sin y$

$= \cos A \csc x \sin (B - x)$

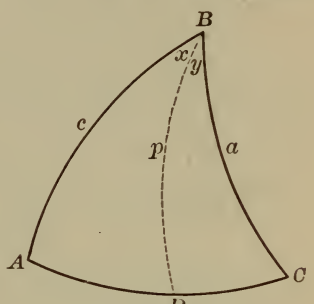


FIG. 101

Now C may be computed from the equations

$$\cot x = \tan A \cos c;$$

$$\cos C = \cos A \csc x \sin (B - x).$$

NOTE. When BD falls to the right of BC, the last equation becomes

$$\cos C = \cos A \csc x \sin (x - B).$$

EXAMPLE. Given $A = 35° 46' 15''$, $B = 115° 9' 7''$, $c = 51° 2'$; find C.

$\log \tan A = 9.85760$	$\log \cos A = 9.90992$
$\log \cos c = 9.79856$	$\log \csc x = 0.04056$
$\log \cot x = 9.65616$	$\log \sin (B - x) = 9.88121$
$x = 65° 37' 35''$	$\log \cos C = 9.83069$
$\therefore B - x = 49° 31' 32''$	$C = 47° 22' 42''$

EXERCISE XXXVIII

1. What are the formulas for computing A when B, C, and a are given; and for computing B when A, C, and b are given?

2. Given $A = 26° 58' 46''$, $B = 39° 45' 10''$, $c = 154° 46' 48''$; find $a = 37° 14' 10''$, $b = 121° 28' 10''$, $C = 161° 22' 11''$.

3. Given $A = 128° 41' 49''$, $B = 107° 33' 20''$, $c = 124° 12' 31''$;
find $a = 125° 41' 43''$, $b = 82° 47' 34''$, $C = 127° 22'$.

4. Given $B = 153° 17' 6''$, $C = 78° 43' 36''$, $a = 86° 15' 15''$;
find $b = 152° 43' 51''$, $c = 88° 12' 21''$, $A = 78° 15' 48''$.

5. Given $A = 125° 41' 44''$, $C = 82° 47' 35''$, $b = 52° 37' 57''$;
find $a = 128° 41' 46''$, $c = 107° 33' 20''$, $B = 55° 47' 40''$.

SECTION LIX

Case III

Given two sides a *and* b, *and the angle* A *opposite* a.

The angle B is found from [44], p. 158; whence we have

$$\sin B = \sin A \sin b \csc a.$$

When B has been found, C and c may be found from the fourth and the second of Napier's Analogies:

$$\tan \tfrac{1}{2} c = \frac{\sin \tfrac{1}{2}(A + B)}{\sin \tfrac{1}{2}(A - B)} \tan \tfrac{1}{2}(a - b);$$

$$\cot \tfrac{1}{2} C = \frac{\sin \tfrac{1}{2}(a + b)}{\sin \tfrac{1}{2}(a - b)} \tan \tfrac{1}{2}(A - B).$$

The third and first of Napier's Analogies may also be used.

Note 1. Since B is determined from its sine, the problem in general has two solutions; and, moreover, in case $\sin B > 1$, the problem is impossible. By geometric construction it may be shown, as in the corresponding case in Plane Trigonometry, pp. 71–73, under what conditions the problem really has two solutions, one solution, or no solution. But in practical applications a general knowledge of the shape of the triangle is known beforehand; so that it is easy to see, without special investigation, which solution (if any) corresponds to the circumstances of the question.

It can be shown that there are *two solutions* when A and a are alike in kind and $\sin b > \sin a > \sin A \sin b$; *no solution* when A and a are unlike in kind (including the case in which either A or a is 90°) and $\sin b$

is greater than $\sin a$ or equal to $\sin a$, or when $\sin a$ is less than $\sin A \sin b$; and *one solution* in every other case.

NOTE 2. The side c or the angle C may be computed, without first finding B, by means of the formulas

$$\tan m = \cos A \tan b, \text{ and } \cos(c - m) = \cos a \sec b \cos m;$$
$$\cot x = \tan A \cos b, \text{ and } \cos(C - x) = \cot a \tan b \cos x.$$

These formulas may be obtained by resolving the triangle into right triangles, and then applying Napier's Rules; m is equal to that part of the side c included between the vertex A and the foot of the perpendicular from C, and x is equal to the corresponding portion of the angle C.

NOTE 3. After the two values of B have been obtained, the number of solutions may be determined by Theorem 1, Sect. XLIX, p. 141.

If $\log \sin B$ is positive, there is no solution.

EXAMPLE. Given $a = 57° 36'$, $b = 31° 14'$, $A = 104° 25' 30''$.

In this case, $A > 90°$,
and $a + b < 180°$.
Therefore, $A + B < 180°$,
and $B < 90°$.
Hence, there is only one solution.

$\log \sin A = 9.98609$
$\log \sin b = 9.71477$
$\log \csc a = \underline{0.07349}$
$\log \sin B = 9.77435$
$B = 36° 29' 46''$

$a + b = 88° 50'$
$a - b = 26° 22'$
$A + B = 140° 55' 16''$
$A - B = 67° 55' 44''$
$\log \sin \frac{1}{2}(A + B) = 9.97424$
$\log \csc \frac{1}{2}(A - B) = 0.25284$
$\log \tan \frac{1}{2}(a - b) = \underline{9.36966}$
$\log \tan \frac{1}{2} c = 9.59674$
$\frac{1}{2} c = 21° 33' 37''$
$c = 43° 7' 14''$

$\frac{1}{2}(a + b) = 44° 25'$
$\frac{1}{2}(a - b) = 13° 11'$
$\frac{1}{2}(A + B) = 70° 27' 38''$
$\frac{1}{2}(A - B) = 33° 57' 52''$
$\log \sin \frac{1}{2}(a + b) = 9.84502$
$\log \csc \frac{1}{2}(a - b) = 0.64194$
$\log \tan \frac{1}{2}(A - B) = \underline{9.82840}$
$\log \cot \frac{1}{2} C = 0.31536$
$\frac{1}{2} C = 25° 48' 58''$
$C = 51° 37' 56''$

EXERCISE XXXIX

1. Given $a = 73° 49' 38''$, $b = 120° 53' 35''$, $A = 88° 52' 42''$;
   find $B = 116° 42' 30''$, $c = 120° 57' 27''$, $C = 116° 47'$.
2. Given $a = 150° 57' 5''$, $b = 134° 15' 54''$, $A = 144° 22' 42''$;
   find $B_1 = 120° 47' 45''$, $c_1 = 55° 42' 8''$, $C_1 = 97° 42' 55''$;
   $B_2 = 59° 12' 15''$, $c_2 = 23° 57' 17''$, $C_2 = 29° 8' 39''$.

3. Given $a = 79° 0' 54''$, $b = 82° 17' 4''$, $A = 82° 9' 26''$;
find $B = 90°$, $c = 45° 12' 19''$, $C = 45° 44' 5''$.

4. Given $a = 30° 52' 37''$, $b = 31° 9' 16''$, $A = 87° 34' 12''$; show that the triangle is impossible.

SECTION LX

Case IV

Given two angles A *and* B, *and the side* a *opposite* A.

The side b is found from [44], p. 158; whence,

$$\sin b = \sin a \sin B \csc A.$$

The values of c and C may then be found by means of the fourth and second of Napier's Analogies:

$$\tan \tfrac{1}{2} c = \frac{\sin \tfrac{1}{2}(A + B)}{\sin \tfrac{1}{2}(A - B)} \tan \tfrac{1}{2}(a - b);$$

$$\cot \tfrac{1}{2} C = \frac{\sin \tfrac{1}{2}(a + b)}{\sin \tfrac{1}{2}(a - b)} \tan \tfrac{1}{2}(A - B).$$

Note 1. In this case the conditions for one solution, two solutions, or no solution can be deduced directly by the theory of polar triangles from the corresponding conditions of Case III, p. 170. There are *two solutions* when A and a are alike in kind and $\sin B > \sin A > \sin a \sin B$; *no solution* when A and a are unlike in kind (including the case in which either A or a is 90°) and $\sin B$ is greater than $\sin A$ or equal to $\sin A$, or when $\sin A < \sin a \sin B$; and *one solution* in every other case.

Note 2. By proceeding as indicated in Case III, Note 2, p. 171, formulas for computing c or C, independent of the side b, may be found; viz.,

$$\tan m = \tan a \cos B, \text{ and } \sin(c - m) = \cot A \tan B \sin m;$$

$$\cot x = \cos a \tan B, \text{ and } \sin(C - x) = \cos A \sec B \sin x.$$

In these formulas $m = BD$, $x = \angle BCD$, D being the foot of the perpendicular from the vertex C.

Note 3. As in Case III, p. 171, only those values of b can be retained which are greater than or less than a, according as B is greater than or less than A. If $\log \sin b$ is positive, the triangle is impossible.

EXERCISE XL

1. Given $A = 110° 10'$, $B = 133° 18'$, $a = 147° 5' 32''$;
   find $b = 155° 5' 18''$, $c = 33° 1' 37''$, $C = 70° 20' 40''$.
2. Given $A = 113° 39' 21''$, $B = 123° 40' 18''$, $a = 65° 39' 46''$;
   find $b = 124° 7' 20''$, $c = 159° 50' 15''$, $C = 159° 43' 34''$.
3. Given $A = 100° 2' 11''$, $B = 98° 30' 28''$, $a = 95° 20' 39''$;
   find $b = 90°$, $c = 147° 41' 50''$, $C = 148° 5' 40''$.
4. Given $A = 24° 33' 9''$, $B = 38° 0' 12''$, $a = 65° 20' 13''$;
show that the triangle is impossible.

SECTION LXI

Case V

Given the three sides a, b, *and* c.

The angles are computed by means of Formulas [47], p. 161, and the corresponding formulas for the angles B and C.

The formulas for the tangent are, in general, to be preferred.

If we multiply the equation

$$\tan \tfrac{1}{2} A = \sqrt{\csc s \csc (s - a) \sin (s - b) \sin (s - c)}$$

by the equation $$1 = \frac{\sin (s - a)}{\sin (s - a)},$$

and put $\tan r$ for $\sqrt{\csc s \sin (s - a) \sin (s - b) \sin (s - c)}$, at the same time making analogous changes in the equations for $\tan \frac{1}{2} B$ and $\tan \frac{1}{2} C$, we obtain

$$\tan \tfrac{1}{2} A = \tan r \csc (s - a),$$
$$\tan \tfrac{1}{2} B = \tan r \csc (s - b),$$
$$\tan \tfrac{1}{2} C = \tan r \csc (s - c),$$

which are the most convenient formulas to employ when all three angles have to be computed.

EXAMPLE 1. Given $a = 50° 54' 32''$, $b = 37° 47' 18''$, $c = 74° 51' 50''$; find A.

$$
\begin{array}{rl}
a = & 50° 54' 32'' \\
b = & 37° 47' 18'' \\
c = & 74° 51' 50'' \\
\hline
2s = & 163° 33' 40'' \\
s = & 81° 46' 50'' \\
s - a = & 30° 52' 18'' \\
s - b = & 43° 59' 32'' \\
s - c = & 6° 55'
\end{array}
\qquad
\begin{array}{rl}
\log \csc s = & 0.00448 \\
\log \csc (s-a) = & 0.28979 \\
\log \sin (s-b) = & 9.84171 \\
\log \sin (s-c) = & 9.08072 \\
\hline
2 \overline{)} & 19.21670 \\
\log \tan \tfrac{1}{2}A = & 9.60835 \\
\tfrac{1}{2}A = & 22° 5' 20'' \\
A = & 44° 10' 40''
\end{array}
$$

EXAMPLE 2. Given $a = 124° 12' 31''$, $b = 54° 18' 16''$, $c = 97° 12' 25''$; find $A = 127° 22' 7''$, $B = 51° 18' 11''$, $C = 72° 26' 40''$.

$$
\begin{array}{rl}
a = & 124° 12' 31'' \\
b = & 54° 18' 16'' \\
c = & 97° 12' 25'' \\
\hline
2s = & 275° 43' 12'' \\
s = & 137° 51' 36'' \\
\log \sin (s-a) = & 9.37293 \\
\log \sin (s-b) = & 9.99725 \\
\log \sin (s-c) = & 9.81390 \\
\log \csc s = & 0.17331 \\
\hline
\log \tan^2 r = & 9.35739 \\
\log \tan r = & 9.67870
\end{array}
\qquad
\begin{array}{rl}
s - a = & 13° 39' 5'' \\
s - b = & 83° 33' 20'' \\
s - c = & 40° 39' 11'' \\
\log \tan \tfrac{1}{2}A = & 0.30577 \\
\log \tan \tfrac{1}{2}B = & 9.68145 \\
\log \tan \tfrac{1}{2}C = & 9.86480 \\
\hline
\tfrac{1}{2}A = & 63° 41' 3.8'' \\
\tfrac{1}{2}B = & 25° 39' 5.6'' \\
\tfrac{1}{2}C = & 36° 13' 20'' \\
\hline
A = & 127° 22' 7'' \\
B = & 51° 18' 11'' \\
C = & 72° 26' 40''
\end{array}
$$

EXERCISE XLI

1. Given $a = 120°55'35''$, $b = 59°4'25''$, $c = 106°10'22''$;
   find $A = 116°44'50''$, $B = 63°15'10''$, $C = 91°7'22''$.
2. Given $a = 50°12'4''$, $b = 116°44'48''$, $c = 129°11'42''$;
   find $A = 59°4'28''$, $B = 94°23'12''$, $C = 120°4'52''$.
3. Given $a = 131°35'4''$, $b = 108°30'14''$, $c = 84°46'34''$;
   find $A = 132°14'21''$, $B = 110°10'40''$, $C = 99°42'24''$.
4. Given $a = 20°16'38''$, $b = 56°19'40''$, $c = 66°20'44''$;
   find $A = 20°9'55''$, $B = 55°52'35''$, $C = 114°20'21''$.

SECTION LXII

Case VI

Given the three angles A, B, *and* C.

The sides are computed by means of Formulas [48], p. 162. The formulas for the tangents are, in general, to be preferred.

If we multiply the equation

$$\tan \tfrac{1}{2} a = \sqrt{-\cos S \cos (S - A) \sec (S - B) \sec (S - C)}$$

by the equation $$1 = \frac{\sec (S - A)}{\sec (S - A)},$$

and put $\tan R$ for $\sqrt{-\cos S \sec (S - A) \sec (S - B) \sec (S - C)}$, at the same time making analogous changes in the equations for $\tan \frac{1}{2} b$ and $\tan \frac{1}{2} c$, we obtain

$$\tan \tfrac{1}{2} a = \tan R \cos (S - A),$$
$$\tan \tfrac{1}{2} b = \tan R \cos (S - B),$$
$$\tan \tfrac{1}{2} c = \tan R \cos (S - C),$$

which are the most convenient formulas to use in case all three sides have to be computed.

Example 1. Given $A = 220°$, $B = 130°$, $C = 150°$; find a.

After we find the values of S, $S - A$, $S - B$, $S - C$, we write the formula for $\tan \frac{1}{2} a$ with the algebraic sign written above each function as follows:

$$\tan \tfrac{1}{2} a = \sqrt{\overset{-}{-\cos} S \overset{+}{\cos} (S - A) \overset{-}{\sec} (S - B) \overset{-}{\sec} (S - C)}.$$

$A = 220°$	$\log \cos S = 9.53405$ (n)
$B = 130°$	$\log \cos (S - A) = 9.93753$
$C = 150°$	$\log \sec (S - B) = 0.30103$ (n)
$2\,S = 500°$	$\log \sec (S - C) = 0.76033$ (n)
$S = 250°$	$2 \overline{) 0.53294}$
$S - A = 30°$	$\log \tan \frac{1}{2} a = 0.26647$
$S - B = 120°$	$\frac{1}{2} a = 61° 34' 6''$
$S - C = 100°$	$a = 123° 8' 12''$

NOTE. Here the effect, as regards algebraic sign, of three negative factors is canceled by the negative sign before the whole product.

EXAMPLE 2. Given $A = 20° 9' 56''$, $B = 55° 52' 32''$, $C = 114° 20' 14''$; find $a = 20° 16' 38''$, $b = 56° 19' 41''$, $c = 66° 20' 43''$.

After we find the values of S, $S - A$, $S - B$, $S - C$, we write the formula for $\tan R$ with the algebraic sign written above each function as follows:

$$\tan R = \sqrt{-\cos S \sec(S - A) \sec(S - B) \sec(S - C)}.$$

$$\begin{aligned} A &= 20° 9' 56'' \\ B &= 55° 52' 32'' \\ C &= 114° 20' 14'' \\ 2S &= 190° 22' 42'' \end{aligned} \qquad \begin{aligned} S &= 95° 11' 21'' \\ S - A &= 75° 1' 25'' \\ S - B &= 39° 18' 49'' \\ S - C &= -19° 8' 53'' \end{aligned}$$

$$\begin{aligned} \log \cos S &= 8.95638 \ (n) \\ \log \sec(S - A) &= 0.58768 \\ \log \sec(S - B) &= 0.11143 \\ \log \sec(S - C) &= 0.02472 \\ \log \tan^2 R &= 9.68021 \\ \log \tan R &= 9.84010 \end{aligned} \qquad \begin{aligned} \log \tan \tfrac{1}{2} a &= 9.25242 \\ \log \tan \tfrac{1}{2} b &= 9.72867 \\ \log \tan \tfrac{1}{2} c &= 9.81538 \\ \tfrac{1}{2} a &= 10° 8' 18.9'' \\ \tfrac{1}{2} b &= 28° 9' 50.3'' \\ \tfrac{1}{2} c &= 33° 10' 21.4'' \\ a &= 20° 16' 38'' \\ b &= 56° 19' 41'' \\ c &= 66° 20' 43'' \end{aligned}$$

EXERCISE XLII

1. Given $A = 130°$, $B = 110°$, $C = 80°$;
   find $a = 139° 21' 22''$, $b = 126° 57' 52''$, $c = 56° 51' 48''$.

2. Given $A = 59° 55' 10''$, $B = 85° 36' 50''$, $C = 59° 55' 10''$;
   find $a = 51° 17' 31''$, $b = 64° 2' 47''$, $c = 51° 17' 31''$.

3. Given $A = 102° 14' 12''$, $B = 54° 32' 24''$, $C = 89° 5' 46''$;
   find $a = 104° 25' 9''$, $b = 53° 49' 25''$, $c = 97° 44' 19''$.

4. Given $A = 4° 23' 35''$, $B = 8° 28' 20''$, $C = 172° 17' 56''$;
   find $a = 31° 9' 13''$, $b = 84° 18' 28''$, $c = 115° 10'$.

SECTION LXIII

AREA OF A SPHERICAL TRIANGLE

I. *When the three angles* A, B, C *are given.*

Let R = the radius of the sphere,

E = the spherical excess $= A + B + C - 180°$,

F = the area of the triangle.

Three planes passed through the centre of a sphere, each perpendicular to the other two planes, divide the surface of the sphere into eight tri-rectangular triangles (*Geometry*, § 802).

It is convenient to divide each of these eight triangles into 90 equal parts, and to call each of these equal parts a spherical degree. The surface of every sphere, therefore, contains 720 spherical degrees.

Now in spherical degrees, the $\triangle ABC = E$ (*Geometry*, § 834), and the surface of the sphere is equal to 720 spherical degrees.

Hence, $\triangle ABC$: surface of the sphere $= E : 720$.

Since the surface of a sphere $= 4\pi R^2$ (*Geometry*, § 824),

$$\triangle ABC : 4\pi R^2 = E : 720.$$

Whence, $$\mathbf{F = \frac{\pi R^2 E}{180}}. \qquad [51]$$

II. *When the three sides* a, b, c *are given.*

A formula for computing the area is deduced as follows: From the first of Formulas [49], p. 164,

$$\frac{\cos \frac{1}{2}(A + B)}{\sin \frac{1}{2} C} = \frac{\cos \frac{1}{2}(a + b)}{\cos \frac{1}{2} c}.$$

Now, $$\sin \tfrac{1}{2} C = \cos (90° - \tfrac{1}{2} C).$$

Therefore, $$\frac{\cos \frac{1}{2}(A + B)}{\cos (90° - \frac{1}{2} C)} = \frac{\cos \frac{1}{2}(a + b)}{\cos \frac{1}{2} c}.$$

Then, by division and composition,

$$\frac{\cos\frac{1}{2}(A+B)-\cos(90^\circ-\frac{1}{2}C)}{\cos\frac{1}{2}(A+B)+\cos(90^\circ-\frac{1}{2}C)}=\frac{\cos\frac{1}{2}(a+b)-\cos\frac{1}{2}c}{\cos\frac{1}{2}(a+b)+\cos\frac{1}{2}c}. \quad \text{(a)}$$

Dividing [23], p. 59, by [22], p. 59,

$$\frac{\cos A-\cos B}{\cos A+\cos B}=-\tan\tfrac{1}{2}(A+B)\tan\tfrac{1}{2}(A-B). \quad \text{(b)}$$

Substituting in (b) for A and B, $\frac{1}{2}(A+B)$ and $90^\circ-\frac{1}{2}C$, respectively, we have

$$\frac{\cos\frac{1}{2}(A+B)-\cos(90^\circ-\frac{1}{2}C)}{\cos\frac{1}{2}(A+B)+\cos(90^\circ-\frac{1}{2}C)}$$

$$=-\tan\tfrac{1}{2}(\tfrac{1}{2}A+\tfrac{1}{2}B+90^\circ-\tfrac{1}{2}C)\tan\tfrac{1}{2}(\tfrac{1}{2}A+\tfrac{1}{2}B-90^\circ+\tfrac{1}{2}C)$$

$$=-\tan\tfrac{1}{4}(A+B-C+180^\circ)\tan\tfrac{1}{4}(A+B+C-180^\circ).$$

Now, $E=A+B+C-180^\circ.$

$$\therefore\ \tan\tfrac{1}{4}(A+B-C+180^\circ)=\tan\tfrac{1}{4}(360^\circ-2C+A+B+C-180^\circ)$$

$$=\tan\tfrac{1}{4}(360^\circ-2C+E)$$

$$=\tan[90^\circ-\tfrac{1}{4}(2C-E)]$$

$$=\cot\tfrac{1}{4}(2C-E).$$

Now, substituting E for $A+B+C-180^\circ$ and $\cot\frac{1}{4}(2C-E)$ for $\tan\frac{1}{4}(A+B-C+180^\circ)$, we have

$$\frac{\cos\frac{1}{2}(A+B)-\cos(90^\circ-\frac{1}{2}C)}{\cos\frac{1}{2}(A+B)+\cos(90^\circ-\frac{1}{2}C)}=-\cot\tfrac{1}{4}(2C-E)\tan\tfrac{1}{4}E. \quad \text{(c)}$$

Again substituting, in equation (b), for A and B the values $\frac{1}{2}(a+b)$ and $\frac{1}{2}c$, and also substitute s for $\frac{1}{2}(a+b+c)$ and $s-c$ for $\frac{1}{2}(a+b-c)$, we have

$$\frac{\cos\frac{1}{2}(a+b)-\cos\frac{1}{2}c}{\cos\frac{1}{2}(a+b)+\cos\frac{1}{2}c}=-\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-c). \quad \text{(d)}$$

Comparing (a), (c), and (d), we obtain

$$\cot\tfrac{1}{4}(2C-E)\tan\tfrac{1}{4}E=\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-c). \quad \text{(e)}$$

By beginning with the second equation of [49], p. 164, and treating it in the same way, we obtain as the result

$$\tan \tfrac{1}{4}(2C - E)\tan \tfrac{1}{4}E = \tan \tfrac{1}{2}(s-a)\tan \tfrac{1}{2}(s-b). \quad \text{(f)}$$

By taking the product of (e) and (f) we obtain the elegant formula, known as l'Huilier's Formula,

$$\tan^2 \tfrac{1}{4}E = \tan \tfrac{1}{2}s \tan \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c). \quad [52]$$

By means of [52] E may be computed from the three sides, and then the area of the triangle may be found by [51], p. 177.

III. *In all other cases*, the area may be found by first solving the triangle so far as to obtain the angles or the sides, whichever may be more convenient, and then applying [51] or [52].

Example 1. Given $A = 102° 14' 12''$, $B = 54° 32' 24''$, $C = 89° 5' 46''$; find the area of the triangle.

$$\begin{aligned} A &= 102° 14' 12'' \\ B &= 54° 32' 24'' \\ C &= 89° 5' 46'' \\ \hline 245° 52' 22'' \\ E &= 65° 52' 22'' \\ &= 237142'' \\ 180° &= 648000'' \end{aligned}$$

$$\begin{aligned} \log R^2 &= \log R^2 \\ \log E &= 5.37501 \\ *\log \frac{\pi}{648000} &= 4.68557 - 10 \\ \hline \log F &= 0.06058 + \log R^2 \\ F &= 1.1497\, R^2 \end{aligned}$$

Hence, if we know the radius of the sphere, we can express the area of a spherical triangle in the ordinary units of area.

Example 2. Given $a = 133° 26' 19''$, $b = 64° 50' 53''$, $c = 144° 13' 45''$; find $E = 200° 46' 46''$.

$$\begin{aligned} a &= 133° 26' 19'' \\ b &= 64° 50' 53'' \\ c &= 144° 13' 45'' \\ \hline 2s &= 342° 30' 57'' \\ s &= 171° 15' 28.5'' \\ s - a &= 37° 49' 9.5'' \\ s - b &= 106° 24' 35.5'' \\ s - c &= 27° 1' 43.5'' \end{aligned}$$

$$\begin{aligned} \log \tan \tfrac{1}{2}s &= 1.11669 \\ \log \tan \tfrac{1}{2}(s-a) &= 9.53474 \\ \log \tan \tfrac{1}{2}(s-b) &= 0.12612 \\ \log \tan \tfrac{1}{2}(s-c) &= 9.38083 \\ \hline \log \tan^2 \tfrac{1}{4}E &= 0.15838 \\ \log \tan \tfrac{1}{4}E &= 0.07919 \\ \tfrac{1}{4}E &= 50° 11' 41.5'' \\ E &= 200° 46' 46'' \end{aligned}$$

* See Wentworth & Hill's Logarithmic and Trigonometric Tables, p. 20.

EXERCISE XLIII

1. Given $A = 84° 20' 19''$, $B = 27° 22' 40''$, $C = 75° 33'$; find $E = 26159''$, $F = 0.12682\ R^2$.

2. Given $a = 69° 15' 6''$, $b = 120° 42' 47''$, $c = 159° 18' 33''$; find $E = 216° 40' 18''$.

3. Given $a = 33° 1' 45''$, $b = 155° 5' 18''$, $C = 110° 10'$; find $E = 133° 48' 53''$.

4. Given $c = 114° 27' 57''$, $A = 78° 42' 33''$, $B = 127° 13' 7''$; find the area.

5. Given $a = 76° 14' 47''$, $b = 82° 40' 15''$, $A = 60° 22' 44''$; find the area.

6. Given $A = 80° 12' 35''$, $B = 77° 38' 22''$, $a = 76° 42' 28''$; find the area.

7. Given $b = 44° 27' 40''$, $c = 15° 22' 44''$, $A = 167° 42' 27''$; find the area.

8. Given $b = 67° 15' 42''$, $A = 84° 55' 8''$, $C = 96° 18' 49''$; find the area.

9. Given $b = 72° 19' 38''$, $c = 54° 58' 52''$, $B = 77° 15' 14''$; find the area.

10. Given $B = 127° 16' 4''$, $C = 42° 34' 19''$, $b = 54° 47' 55''$; find the area.

11. Given $a = 128° 42' 56''$, $b = 107° 13' 48''$, $c = 88° 37' 51''$; find the area.

12. Given $A = 127° 22' 28''$, $B = 131° 45' 27''$, $C = 100° 52' 16''$; find the area.

13. Given $a = 116° 19' 45''$, $A = 160° 42' 24''$, $C = 171° 27' 15''$; find the area.

14. Find the area of a triangle on the earth's surface (regarded as spherical) if each side of the triangle is equal to 1°. (Radius of earth = 3958 miles.)

CHAPTER IX

APPLICATIONS OF SPHERICAL TRIGONOMETRY

SECTION LXIV

PROBLEM

To reduce to the horizon an angle measured in space.

Let O (Fig. 102) be the position of the eye; $DOC = h$, the angle measured in space; OD' and OC' the projections of the sides of the angle upon the horizontal plane; $DOD' = m$ and

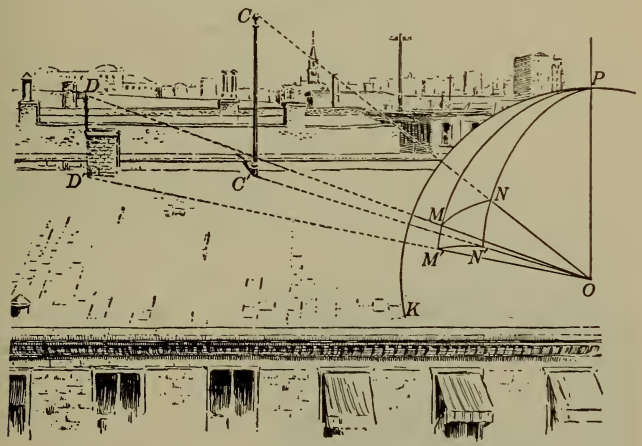


Fig. 102

$COC' = n$, the angles of inclination of OD and OC, respectively, to the horizon. Required the angle $D'OC' = x$ made by the projections on the horizon.

The planes of the angles DOD' and COC' intersect in the line OP, perpendicular to the horizontal plane (Wentworth's *Geometry*, § 556).

From O as a centre describe a sphere, and let its surface cut the edges of the trihedral angle O–DCP in M, N, and P.

In the spherical triangle MNP, $MN = h$, $MP = 90° - m$, $NP = 90° - n$ are known; and $P = x$ is required.

By [47], p. 161,

$$\cos \tfrac{1}{2}x = \sqrt{\frac{\sin(90° + \tfrac{1}{2}h - \tfrac{1}{2}m - \tfrac{1}{2}n)\sin(90° - \tfrac{1}{2}h - \tfrac{1}{2}m - \tfrac{1}{2}n)}{\sin(90° - m)\sin(90° - n)}}.$$

Putting $\frac{1}{2}(h + m + n) = s$, we obtain

$$\cos \tfrac{1}{2}x = \sqrt{\frac{\sin[90° + (s - h)]\sin(90° - s)}{\cos m \cos n}}$$

$$= \sqrt{\cos(s - h)\cos s \sec m \sec n}.$$

SECTION LXV

PROBLEM

To find the distance between two places on the earth's surface (regarded as spherical), given the latitudes of the places and the difference of their longitudes.

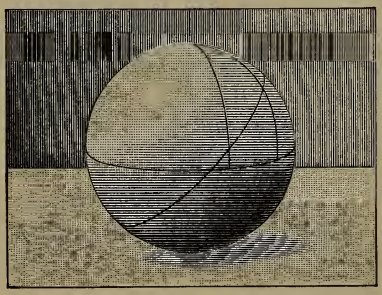


FIG. 103

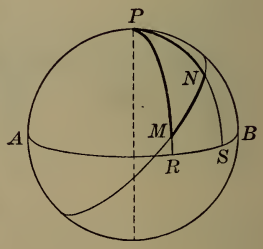


FIG. 104

Let M and N (Fig. 104) be the places. Then the distance MN is an arc of the great circle passing through the places. Let P be the pole, ARB the equator. The arcs MR and NS are the latitudes of the places, and the arc RS, or the angle MPN, is the difference of their longitudes. Let $MR = b$, $NS = a$, $RS = l$; then in the spherical triangle MNP two sides, $MP = 90° - b$, $NP = 90° - a$, and the included angle $MPN = l$ are given, and we have from Sect. LVII, p. 167,

$$\tan m = \cot a \cos l,$$
$$\cos MN = \sin a \sec m \sin (b + m).$$

From these equations first find m, then the arc MN, and then reduce MN to geographical miles, of which there are 60 in each degree.

SECTION LXVI

THE CELESTIAL SPHERE

The **Celestial Sphere** is an imaginary sphere of indefinite radius, upon the concave surface of which all the heavenly bodies appear to be situated.

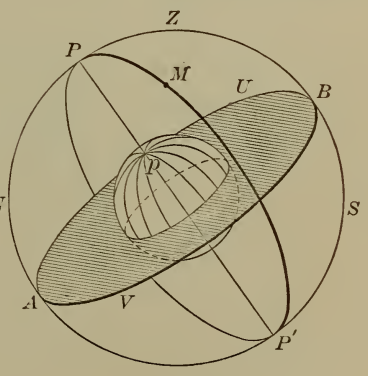


FIG. 105

The **Celestial Equator**, or **Equinoctial**, $AVBU$ (Fig. 105), is the great circle in which the plane of the earth's equator intersects the surface of the celestial sphere.

The **Poles**, P and P' (Fig. 105), of the celestial equator are the points in which the earth's axis produced cuts the surface of the celestial sphere.

The **Celestial Meridian**, PBP' (Fig. 106), of an observer is the great circle in which the plane of his terrestrial meridian produced meets the surface of the celestial sphere.

Hour Circles, as PMP' (Fig. 106), or **Circles of Declination**, are great circles passing through the poles, perpendicular to the celestial equator.

The **Horizon**, $NWSE$ (Fig. 106), of an observer is the great circle in which the plane tangent to the earth's surface, at the place where he is, meets the surface of the celestial sphere.

The **Zenith**, Z (Fig. 106), of an observer is that pole of his horizon which is exactly above his head.

Vertical Circles, as $NPZS$ (Fig. 106), are great circles passing through the zenith of an observer, perpendicular to his horizon.

The vertical circle passing through the east and west points of the horizon is called the **Prime Vertical**; that passing through the north and south points coincides with the celestial meridian.

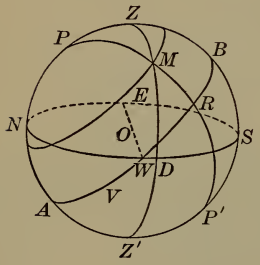


FIG. 106

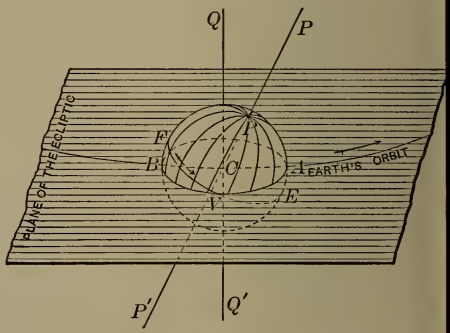


FIG. 107

The **Ecliptic**, AVB (Fig. 107), is a great circle of the celestial sphere, apparently traversed by the sun in one year from west to east, in consequence of the motion of the earth around the sun.

The **Equinoxes** are the points where the ecliptic cuts the celestial equator. They are distinguished as the *Vernal* equinox and the *Autumnal* equinox; the sun in his annual journey passes through the former on March 21, and through the latter on September 21. The Vernal equinox is shown here at V (Fig. 108).

Circles of Latitude, as QMT (Fig. 108), are great circles passing through the poles of the ecliptic, perpendicular to the plane of the ecliptic.

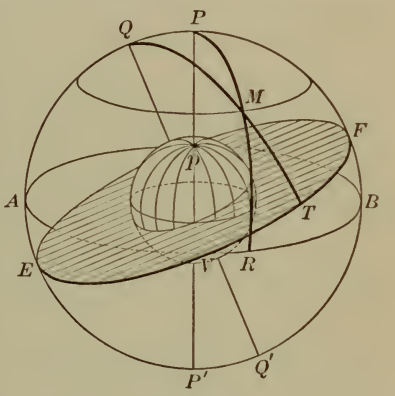


FIG. 108

The angle which the ecliptic makes with the celestial equator is called the **obliquity** of the ecliptic; it is equal to 23° 27′, nearly, and is often denoted by the letter *e*.

The earth's diurnal motion causes all the heavenly bodies to appear to rotate from east to west at the uniform rate of 15° per hour. If in Fig. 106 we conceive the observer placed at the centre O, and his zenith, horizon, and celestial meridian fixed in position, and all the heavenly bodies rotating from east to west around PP' as an axis at the rate of 15° per hour, we form a correct idea of the apparent diurnal motions of these bodies. When the sun or a star in its diurnal motion crosses the meridian, it is said to make a *transit* across the meridian; when it passes across the part NWS of the horizon, it is said to *set;* and when it passes across the part NES, it is said to *rise* (the effect of refraction being here neglected). Each star, as M, describes daily a small circle of the sphere parallel to the celestial equator, and called the **diurnal circle** of the star. The nearer the star is to

the pole, the smaller is the diurnal circle; and if there were stars at the poles P and P', they would have no diurnal motion. To an observer north of the equator the north pole P is *elevated* above the horizon (Fig. 108); to an observer south of the equator the south pole P' is the elevated pole.

SECTION LXVII

SPHERICAL CO-ORDINATES

Several systems of fixing the position of a star on the surface of the celestial sphere at any instant are in use. In each system a great circle and its pole are taken as standards of reference, and the position of the star is determined by means of two quantities called its *spherical co-ordinates.*

I. *If the horizon and the zenith* are chosen (Fig. 109), the co-ordinates of the star are called its altitude and its azimuth.

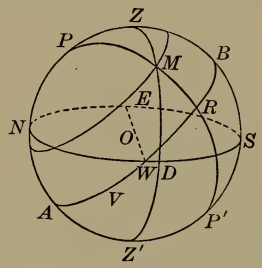


FIG. 109

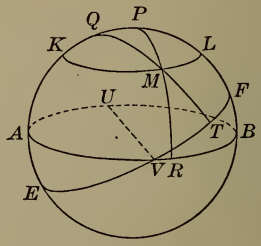


FIG. 110

The **Altitude** of a star is its angular distance, DM, measured on a vertical circle, above the horizon. The complement, MZ, of the altitude is called the **Zenith Distance.**

The **Azimuth** of a star is the angle PZM at the zenith formed by the meridian of the observer and the vertical circle passing through the star, and is measured, therefore, by an arc of the horizon. It is usually reckoned from the north point of the

horizon in north latitudes, and from the south point in south latitudes; and east or west according as the star is east or west of the meridian.

II. *If the celestial equator and its pole* are chosen (Fig. 110), then the position of the star may be fixed by means of its declination and its hour angle.

The **Declination** of a star is its angular distance, RM, from the celestial equator, measured on an hour circle. The angular distance, PM, of the star, measured on the hour circle, *from the elevated pole*, is called its **Polar Distance**.

The declination of a star, like the latitude of a place on the earth's surface, may be either north or south; but, in practical problems, while latitude is always to be considered positive, declination, if of a different name from the latitude, must be regarded as *negative*.

If the declination is negative, the polar distance is equal numerically to 90° + the declination.

The **Hour Angle** of a star is the angle MPQ at the pole formed by the meridian, APB, of the observer and the hour circle, PMR, passing through the star. On account of the diurnal rotation the hour angle is constantly changing at the rate of 15° per hour. Hour angles are reckoned from the celestial meridian, positive towards the west, and negative towards the east.

III. *The celestial equator and its pole* being still retained, we may employ as the co-ordinates of the star its declination and its right ascension.

The **Right Ascension** of a star is the arc VR of the celestial equator included between the Vernal equinox and the point where the hour circle of the star cuts the celestial equator. Right ascension is reckoned from the Vernal equinox *eastward* from 0° to 360°.

IV. *The ecliptic, EVF, and its pole Q* may be taken as the standards of reference. The co-ordinates of the star are then called its *latitude* and its *longitude*.

The **Latitude** of a star is its angular distance, MT (Fig. 112), from the ecliptic measured on a circle of latitude.

The **Longitude** of a star is the arc VT (Fig. 112) of the ecliptic included between the Vernal equinox and the point where the circle of latitude through the star cuts the ecliptic. The longitude of a star is always measured *eastward* from the Vernal equinox.

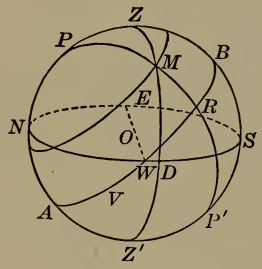


FIG. 111

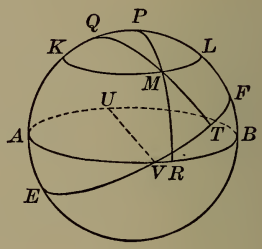


FIG. 112

For the star M (Figs. 111 and 112), let

$l =$ the latitude of the observer,

$h =$ $DM =$ the altitude of the star (Fig. 111),

$z =$ $ZM =$ the zenith distance of the star,

$a = \angle PZM =$ the azimuth of the star,

$t = \angle ZPM =$ the hour angle of the star,

$d =$ $RM =$ the declination of the star,

$p =$ $PM =$ the polar distance of the star,

$r =$ $VR =$ the right ascension of the star (Fig. 112),

$u =$ $MT =$ the latitude of the star,

$v =$ $VT =$ the longitude of the star,

$NZS =$ the celestial meridian (Fig. 111),

$ARB =$ the celestial equator (Fig. 112),

$EVF =$ the ecliptic.

In many problems a simple way of representing the magnitudes involved is to project the sphere on the plane of the horizon, as shown in Figs. 114 and 115.

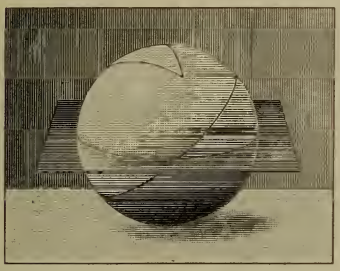


FIG. 113

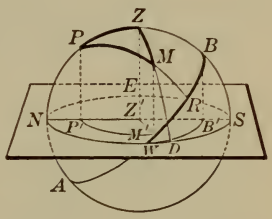


FIG. 114

$NESW$ is the horizon, Z the zenith, NZS the celestial meridian, WZE the prime vertical, WBE an arc of the celestial equator, P the pole, M a star, DM its altitude, ZM its zenith distance, $\angle PZM$ its azimuth, MR its declination, PM its polar distance, $\angle ZPM$ its hour angle.

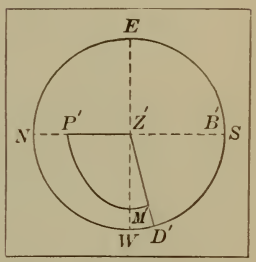


FIG. 115

SECTION LXVIII

THE ASTRONOMICAL TRIANGLE

The triangle ZPM (Fig. 111) is often called the *astronomical triangle*, on account of its importance in problems in Nautical Astronomy.

The side PZ is equal to the complement of the latitude of the observer. For (Fig. 111) since O is the centre of the sphere, the angle ZOB between the zenith of the observer and the celestial equator is obviously equal to his latitude,

and the angle POZ is the complement of ZOB. Since the arc NP is the complement of PZ, *the altitude of the elevated pole is equal to the latitude of the place of observation.*

The triangle ZPM then (however much it may vary in shape for different positions of the star M) always contains the following five magnitudes:

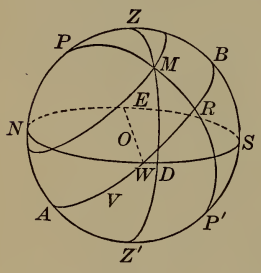


FIG. 116

PZ = the co-latitude of observer
$= 90° - l$,

ZM = the zenith distance of star
$= z$,

$\angle PZM$ = the azimuth of star
$= a$,

PM = the polar distance of star
$= p$,

$\angle ZPM$ = the hour angle of star
$= t$.

A very simple relation exists between the hour angle of the sun and the local (apparent) time of day. The hourly rate at which the sun appears to move from east to west is 15°, and it is apparent noon at any place when the sun is on the meridian of that place. Hence, it is evident that if the hour angle is 0°, the time of day is noon; if the hour angle is 15°, the time of day is 1 o'clock P.M.; if the hour angle is 75°, the time of day is 5 o'clock P.M.; if the hour angle is $-15°$, the time of day is 11 o'clock A.M.; if the hour angle is $-75°$, the time of day is 7 o'clock A.M.; and so on.

In general, if t denotes the absolute value of the hour angle, when the sun is west of the meridian,

$$\text{the time of day is } \frac{t}{15} \text{ P.M.};$$

when the sun is east of the meridian,

$$\text{the time of day is } 12 - \frac{t}{15} \text{ A.M.}$$

SECTION LXIX

PROBLEM

Given the latitude of the observer and the altitude and the azimuth of a star, to find its declination and its hour angle.

In the triangle ZPM (Fig. 117), given

$PZ = 90° - l$ = the co-latitude,

$ZM = 90° - h$ = the co-altitude,

$\angle PZM = a$ = the azimuth;

to find

$PM = 90° - d$ = the polar distance,

$\angle ZPM = t$ = the hour angle.

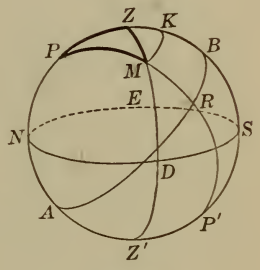


FIG. 117

Draw $MK \perp$ to NZS.

Let $ZK = m$.

Then, if $a < 90°$, $PK = 90° - (l + m)$;

and if $a > 90°$, $PK = 90° - (l - m)$.

By Napier's Rules,

$$\cos a = \pm \tan m \tan h, \tag{1}$$

$$\sin d = \cos PK \cos MK, \tag{2}$$

$$\sin h = \cos m \cos MK. \tag{3}$$

From (1), $\tan m = \pm \cot h \cos a.$ (**A**)

From (3), $\cos MK = \sin h \sec m.$ (4)

From (2), $\sin d = \cos [90° - (l \pm m)] \cos MK.$

$$\therefore \sin d = \sin (l \pm m) \cos MK. \tag{5}$$

Substitute in (5) the value of $\cos MK$.

Then, $\sin d = \sin (l \pm m) \sin h \sec m.$ (**B**)

In equations (**A**) and (**B**) the $-$ sign is to be used if $a > 90°$. The hour angle may then be found by means of [44], p. 158, whence $\sin t = \sin a \cos h \sec d.$

SECTION LXX

PROBLEM

To find the hour angle of a heavenly body when its declination, its altitude, and the latitude of the place are known.

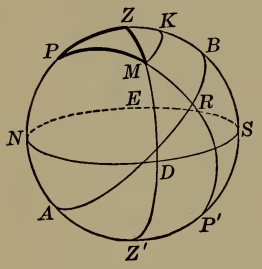


FIG. 118

In the triangle ZPM (Fig. 118),

given $PZ = 90° - l,$

$PM = 90° - d = p,$

$ZM = 90° - h;$

required

$\angle ZPM = t.$

If, in the first formula of [47], p. 161,

$$\sin \tfrac{1}{2} A = \sqrt{\sin(s-b)\sin(s-c)\csc b \csc c},$$

we put $A = t$, $a = 90° - h$, $b = p$, $c = 90° - l$,

and $2s = a + b + c.$

Then, $2s = 90° - h + p + 90° - l,$

or $2s = 180° - l + p - h;$

whence, $s = 90° - \tfrac{1}{2} l + \tfrac{1}{2} p - \tfrac{1}{2} h,$

$s - b = 90° - \tfrac{1}{2}(l + p + h),$

$s - c = \tfrac{1}{2}(l + p - h);$

and the formula becomes

$$\sin \tfrac{1}{2} t = \pm \sqrt{\sin[90° - \tfrac{1}{2}(l+p+h)] \sin \tfrac{1}{2}(l+p-h) \csc p \csc(90° - l)}$$

$$= \pm \sqrt{\cos \tfrac{1}{2}(l+p+h) \sin \tfrac{1}{2}(l+p-h) \csc p \sec l},$$

in which the $-$ sign is to be taken when the body is east of the meridian.

If the body is the sun, how can the local time be found when the hour angle has been computed (Sect. LXVIII, p. 189)?

SECTION LXXI

PROBLEM

To find the altitude and the azimuth of a celestial body when its declination, its hour angle, and the latitude of the place are known.

In the triangle ZPM (Fig. 118),

given $PZ = 90° - l,$

$PM = 90° - d = p,$

$\angle ZPM = t;$

required $ZM = 90° - h,$

$\angle ZPM = a.$

Draw $MK \perp$ to NZS and let $PK = m.$

Then, if $a < 90°$, $ZK = 90° - (l + m);$

and if $a > 90°$, $ZK = (l + m) - 90°.$

By Napier's Rules,

$$\cos t = \tan m \tan d, \tag{1}$$

$$\sin h = \sin (l + m) \cos MK, \tag{2}$$

$$\sin d = \cos MK \cos m, \tag{3}$$

$$\cos (l + m) = \cot a \tan MK, \tag{4}$$

$$\sin m = \cot t \tan MK. \tag{5}$$

From (1), $\tan m = \cot d \cos t.$ **(A)**

From (3), $\cos MK = \sin d \sec m.$ (6)

Substitute in (2) the value of cos MK,

$$\sin h = \sin (l + m) \sin d \sec m. \tag{B}$$

From (5), $\tan MK = \sin m \tan t.$

Substitute in (4) the value of tan MK,

$$\tan a = \sec (l + m) \sin m \tan t. \tag{C}$$

Equations **(A)**, **(B)**, and **(C)** are the equations required.

In **(C)** a is E. or W. to agree with the hour angle.

SECTION LXXII

PROBLEM

To find the latitude of the place when the altitude of a celestial body, its declination, and its hour angle are known.

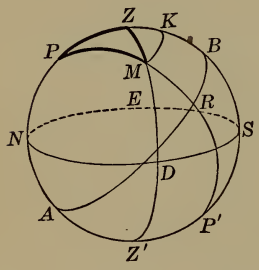


FIG. 119

In the triangle ZPM (Fig. 119),

given $ZM = 90° - h$,

$PM = 90° - d$,

$\angle ZPM = t$;

required $PZ = 90° - l$.

Draw $MK \perp$ to NZS.

Let $PK = m$, $ZK = n$.

Then, by Napier's Rules,

$$\cos t = \tan m \tan d,$$

$$\sin h = \cos n \cos MK,$$

$$\sin d = \cos m \cos MK;$$

whence,

$$\tan m = \cot d \cos t,$$

$$\cos n = \cos m \sin h \csc d.$$

It is evident from the figure that

$$l = 90° - (m \pm n),$$

in which the sign + or the sign − is to be taken according as the celestial body and the elevated pole are *on the same side* of the prime vertical or *on opposite sides*.

In fact, both the values of l shown above may be possible for the same altitude and the same hour angle; but, unless n is very small, the two values will differ largely from each other, so that the observer has no difficulty in deciding which of them should be taken.

SECTION LXXIII

PROBLEM

Given the declination, the right ascension of a star, and the obliquity of the ecliptic, to find the latitude and the longitude of the star.

Let M (Fig. 121) be the star, P the pole of the celestial equator, and Q the pole of the ecliptic.

Then, in the triangle PMQ,

given $PQ = e = 23° 27'$,

$PM = 90° - d$,

$\angle MPQ = 90° + r$;

required $QM = 90° - u$,

and $\angle PQM = 90° - v$;

where $r =$ the right ascension $= VR$,

$u =$ the latitude of $M = MT$,

and $v =$ the longitude of $M = VT$.

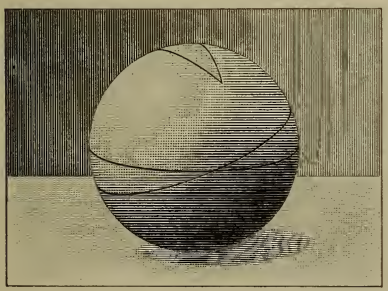


FIG. 120

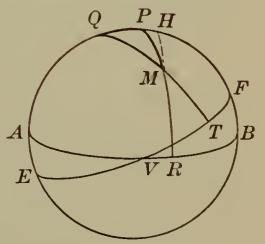


FIG. 121

Here, two sides and the included angle are given. Draw $MH \perp$ to PQ, and meeting it produced at H. Let $PH = n$.

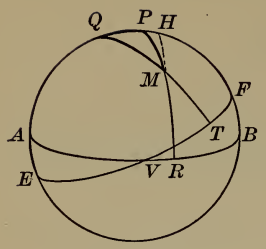


FIG. 122

By Napier's Rules,

$$\sin r = \tan n \tan d, \tag{1}$$

$$\sin u = \cos(e+n)\cos MH, \tag{2}$$

$$\sin d = \cos n \cos MH, \tag{3}$$

$$\sin(e+n) = \tan v \tan MH, \tag{4}$$

$$\sin n = \tan r \tan MH. \tag{5}$$

From (1),

$$\tan n = \cot d \sin r. \tag{A}$$

From (3), $\cos MH = \sin d \sec n.$

Substitute in (2) the value of $\cos MH$,

$$\sin u = \cos(e+n)\sin d \sec n. \tag{B}$$

From (4), $$\tan v = \sin(e+n)\cot MH. \tag{6}$$

From (5), $\cot MH = \tan r \csc n.$

Substitute in (6) the value of $\cot MH$,

$$\tan v = \sin(e+n)\tan r \csc n. \tag{C}$$

Equations (**A**), (**B**), and (**C**) determine u and v.

To avoid obtaining u from its sine we may proceed as follows:

From equations (**B**) and (**C**) we have, by division,

$$\frac{\sin u}{\tan v} = \frac{\sin d \cos(e+n)\sec n}{\tan r \sin(e+n)\csc n}.$$

$$\therefore \sin u = \tan v \sin d \cot r \cot(e+n)\tan n.$$

By taking MH as middle part, successively, in the triangles MQH and MPH, we obtain

$$\cos MH = \cos u \cos v,$$

and $$\cos MH = \cos d \cos r.$$

$$\therefore \cos u \cos v = \cos d \cos r.$$

$$\therefore \cos u = \sec v \cos d \cos r.$$

From these values of $\sin u$ and $\cos u$ we obtain, by division,

$$\frac{\sin u}{\cos u} = \frac{\tan v \cot (e + n) \sin d \cot r \tan n}{\sec v \cos d \cos r}.$$

$$\therefore \tan u = \sin v \cot (e + n) \tan d \csc r \tan n.$$

From the relation

$$\sin r = \tan n \tan d,$$

it follows that, dividing by $\sin r$,

$$\tan d \csc r \tan n = 1.$$

Therefore, $$\tan u = \sin v \cot (e + n),$$

a formula by which u can be easily found after v has been computed.

EXERCISE XLIV

1. Find the dihedral angle made by adjacent lateral faces of a regular ten-sided pyramid; given the angle $V = 18°$, made at the vertex by two adjacent lateral edges.

2. Through the foot of a rod which makes the angle A with a plane a straight line is drawn in the plane. This line makes the angle B with the projection of the rod upon the plane. What angle does this line make with the rod?

3. Find the volume V of an oblique parallelopipedon; given the three unequal edges a, b, c, and the three angles l, m, n, which the edges make with one another.

4. The continent of Asia has nearly the shape of an equilateral triangle, the vertices being the East Cape, Cape Romania, and the Promontory of Baba. Assuming each side of this triangle to be 4800 geographical miles, and the earth's radius to be 3440 geographical miles, find the area of the triangle: (i) regarded as a plane triangle; (ii) regarded as a spherical triangle.

5. A ship sails from a harbor in latitude l and keeps on the arc of a great circle. Her *course* (or angle between the direction

in which she sails and the meridian) at starting is a. Find where she will cross the equator, her course at the equator, and the distance she has sailed.

6. Two places have the same latitude l, and the distance between the places, measured on an arc of a great circle, is d. How much greater is the arc of the parallel of latitude between the places than the arc of the great circle? Compute the results for $l = 45°$, $d = 90°$.

7. The distance d between two places and the latitudes l and l' of the places are known. Find the difference between their longitudes.

8. Given the latitudes and longitudes of three places on the earth's surface, and also the radius of the earth; show how to find the area of the spherical triangle formed by arcs of great circles passing through the three places.

9. The distance between Paris and Berlin (the arc of a great circle) is equal to 472 geographical miles. The latitude of Paris is 48° 50′ 13″; that of Berlin, 52° 30′ 16″. When it is noon at Paris what time is it at Berlin?

NOTE. Owing to the apparent motion of the sun, the local time over the earth's surface at any instant varies at the rate of one hour for 15° of longitude; and the more *easterly* the place, the *later* the local time.

10. Given the altitude of the pole 45°, and the azimuth of a star on the horizon 45°; find the polar distance of the star.

11. Given the latitude l of the observer, and the declination d of the sun; find the local time (apparent solar time) of sunrise and sunset, and also the azimuth of the sun at these times (refraction being neglected). When and where does the sun rise on the longest day of the year (at which time $d = +23° 27'$) in Boston ($l = 42° 21'$), and what is the length of the day from sunrise to sunset? Also, find when and where the sun rises

in Boston on the shortest day of the year (when $d = -23° 27'$), and the length of this day.

12. When is the solution of the problem in Example 11 impossible, and for what places is the solution impossible?

13. Given the latitude of a place and the sun's declination; find his altitude and azimuth at 6 o'clock A.M. (neglecting refraction). Compute the results for the longest day of the year at Munich ($l = 48° 9'$).

14. How does the altitude of the sun at 6 A.M. *on a given day* change as we go from the equator to the pole? At what time of the year is it a maximum *at a given place?* (Given $\sin h = \sin l \sin d$.)

15. Given the latitude of a place north of the equator, and the declination of the sun; find the time of day when the sun bears due east and due west. Compute the results for the longest day at St. Petersburg ($l = 59° 56'$).

16. Apply the general result in Example 15 ($\cos t = \cot l \tan d$) to the case when the days and nights are equal in length (that is, when $d = 0°$). Why can the sun in summer never be due east before 6 A.M., or due west after 6 P.M.? How does the time of bearing due east and due west change with the declination of the sun? Apply the general result to the cases where $l < d$ and $l = d$. What is it at the north pole?

17. Given the sun's declination and his altitude when he bears due east; find the latitude of the observer.

18. At a point O in a horizontal plane MN a staff OA is fixed so that its angle of inclination AOB with the plane is equal to the latitude of the place, 51° 30′ N., and the direction OB is due north. What angle will OB make with the shadow of OA on the plane, at 1 P.M., when the sun is on the equinoctial?

19. What is the direction of a wall in latitude 52° 30′ N. which casts no shadow at 6 A.M. on the longest day of the year?

20. Find the latitude of the place at which the sun rises exactly in the northeast on the longest day of the year.

21. Find the latitude of the place at which the sun sets at 10 o'clock on the longest day.

22. To what does the general formula for the hour angle, in Sect. LXX, reduce when (i) $h = 0°$, (ii) $l = 0°$ and $d = 0°$, (iii) l or $d = 90°$?

23. What does the general formula for the azimuth of a celestial body, in Sect. LXXI, become when $t = 90° = 6$ hours?

24. Show that the formulas of Sect. LXXII, if $t = 90°$, lead to the equation $\sin l = \sin h \csc d$; and that if $d = 0°$, they lead to the equation $\cos l = \sin h \sec t$.

25. Given the latitude of the place of observation $52°\,30'\,16''$, the declination of a star 38°, its hour angle $28°\,17'\,15''$; find the altitude of the star.

26. Given the latitude of the place of observation $51°\,19'\,20''$, the polar distance of a star $67°\,59'\,5''$, its hour angle $15°\,8'\,12''$; find the altitude and the azimuth of the star.

27. Given the declination of a star $7°\,54'$, its altitude $22°\,45'\,12''$, its azimuth $129°\,45'\,37''$; find the hour angle of the star and the latitude of the observer.

28. Given $e = 23°\,27'$ and the longitude v of the sun; find the declination d and the right ascension r.

29. Given $e = 23°\,27'$, the latitude of a star 51°, its longitude 315°; find its declination and its right ascension.

30. Given the latitude of the observer $44°\,50'$, the azimuth of a star $138°\,58'$, its hour angle 20°; find its declination.

31. Given the latitude of the place of observation $51°\,31'\,48''$, the altitude of the sun west of the meridian $35°\,14'\,27''$, its declination $+\,21°\,27'$; find the local apparent time.

32. Given the latitude of a place l, the polar distance p of a star, and its altitude h; find its azimuth a.

FORMULAS

PLANE TRIGONOMETRY

1. $\sin^2 A + \cos^2 A = 1.$

2. $\tan A = \dfrac{\sin A}{\cos A}.$

3. $\begin{cases} \sin A \times \csc A = 1. \\ \cos A \times \sec A = 1. \\ \tan A \times \cot A = 1. \end{cases}$

4. $\sin (x + y) = \sin x \cos y + \cos x \sin y.$

5. $\cos (x + y) = \cos x \cos y - \sin x \sin y.$

6. $\tan (x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}.$

7. $\cot (x + y) = \dfrac{\cot x \cot y - 1}{\cot y + \cot x}.$

8. $\sin (x - y) = \sin x \cos y - \cos x \sin y.$

9. $\cos (x - y) = \cos x \cos y + \sin x \sin y.$

10. $\tan (x - y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}.$

11. $\cot (x - y) = \dfrac{\cot x \cot y + 1}{\cot y - \cot x}.$

12. $\sin 2x = 2 \sin x \cos x.$

13. $\cos 2x = \cos^2 x - \sin^2 x.$

14. $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}.$

15. $\cot 2x = \dfrac{\cot^2 x - 1}{2 \cot x}$.

16. $\sin \frac{1}{2} z = \pm \sqrt{\dfrac{1 - \cos z}{2}}$.

17. $\cos \frac{1}{2} z = \pm \sqrt{\dfrac{1 + \cos z}{2}}$.

18. $\tan \frac{1}{2} z = \pm \sqrt{\dfrac{1 - \cos z}{1 + \cos z}}$.

19. $\cot \frac{1}{2} z = \pm \sqrt{\dfrac{1 + \cos z}{1 - \cos z}}$.

20. $\sin A + \sin B = 2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)$.

21. $\sin A - \sin B = 2 \cos \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)$.

22. $\cos A + \cos B = 2 \cos \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)$.

23. $\cos A - \cos B = -2 \sin \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)$.

24. $\dfrac{\sin A + \sin B}{\sin A - \sin B} = \dfrac{\tan \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A - B)}$.

25. $\dfrac{a}{b} = \dfrac{\sin A}{\sin B}$.

26. $a^2 = b^2 + c^2 - 2bc \cos A$.

27. $\dfrac{a - b}{a + b} = \dfrac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)}$.

28. $\sin \frac{1}{2} A = \sqrt{\dfrac{(s - b)(s - c)}{bc}}$.

29. $\cos \frac{1}{2} A = \sqrt{\dfrac{s(s - a)}{bc}}$.

30. $\tan \frac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$

31. $\sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = r.$

32. $\tan \frac{1}{2} A = \frac{r}{s-a}.$

33. $F = \frac{1}{2} ac \sin B.$

34. $F = \frac{a^2 \sin B \sin C}{2 \sin (B+C)}.$

35. $F = \sqrt{s(s-a)(s-b)(s-c)}.$

36. $F = \frac{abc}{4R}.$

37. $F = \frac{1}{2} r(a+b+c) = rs.$

SPHERICAL TRIGONOMETRY

38. $\cos c = \cos a \cos b.$

39. $\begin{cases} \sin a = \sin c \sin A. \\ \sin b = \sin c \sin B. \end{cases}$

40. $\begin{cases} \cos A = \tan b \cot c. \\ \cos B = \tan a \cot c. \end{cases}$

41. $\begin{cases} \cos A = \cos a \sin B. \\ \cos B = \cos b \sin A. \end{cases}$

42. $\begin{cases} \sin b = \tan a \cot A. \\ \sin a = \tan b \cot B. \end{cases}$

43. $\cos c = \cot A \cot B.$

44. $\begin{cases} \sin a \sin B = \sin b \sin A. \\ \sin a \sin C = \sin c \sin A. \\ \sin b \sin C = \sin c \sin B. \end{cases}$

45. $$\begin{cases} \cos a = \cos b \cos c + \sin b \sin c \cos A. \\ \cos b = \cos a \cos c + \sin a \sin c \cos B. \\ \cos c = \cos a \cos b + \sin a \sin b \cos C. \end{cases}$$

46. $$\begin{cases} \cos A = -\cos B \cos C + \sin B \sin C \cos a. \\ \cos B = -\cos A \cos C + \sin A \sin C \cos b. \\ \cos C = -\cos A \cos B + \sin A \sin B \cos c. \end{cases}$$

47. $$\begin{cases} \sin \tfrac{1}{2} A = \sqrt{\sin (s-b) \sin (s-c) \csc b \csc c}. \\ \cos \tfrac{1}{2} A = \sqrt{\sin s \sin (s-a) \csc b \csc c}. \\ \tan \tfrac{1}{2} A = \sqrt{\csc s \csc (s-a) \sin (s-b) \sin (s-c)}. \end{cases}$$

48. $$\begin{cases} \sin \tfrac{1}{2} a = \sqrt{-\cos S \cos (S-A) \csc B \csc C}. \\ \cos \tfrac{1}{2} a = \sqrt{\cos (S-B) \cos (S-C) \csc B \csc C}. \\ \tan \tfrac{1}{2} a = \sqrt{-\cos S \cos (S-A) \sec (S-B) \sec (S-C)}. \end{cases}$$

49. $$\begin{cases} \cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2} c = \cos \tfrac{1}{2}(a+b) \sin \tfrac{1}{2} C. \\ \sin \tfrac{1}{2}(A+B) \cos \tfrac{1}{2} c = \cos \tfrac{1}{2}(a-b) \cos \tfrac{1}{2} C. \\ \cos \tfrac{1}{2}(A-B) \sin \tfrac{1}{2} c = \sin \tfrac{1}{2}(a+b) \sin \tfrac{1}{2} C. \\ \sin \tfrac{1}{2}(A-B) \sin \tfrac{1}{2} c = \sin \tfrac{1}{2}(a-b) \cos \tfrac{1}{2} C. \end{cases}$$

50. $$\begin{cases} \tan \tfrac{1}{2}(A+B) = \dfrac{\cos \tfrac{1}{2}(a-b)}{\cos \tfrac{1}{2}(a+b)} \cot \tfrac{1}{2} C. \\ \tan \tfrac{1}{2}(A-B) = \dfrac{\sin \tfrac{1}{2}(a-b)}{\sin \tfrac{1}{2}(a+b)} \cot \tfrac{1}{2} C. \\ \tan \tfrac{1}{2}(a+b) = \dfrac{\cos \tfrac{1}{2}(A-B)}{\cos \tfrac{1}{2}(A+B)} \tan \tfrac{1}{2} c. \\ \tan \tfrac{1}{2}(a-b) = \dfrac{\sin \tfrac{1}{2}(A-B)}{\sin \tfrac{1}{2}(A+B)} \tan \tfrac{1}{2} c. \end{cases}$$

51. $$F = \frac{\pi R^2 E}{180}.$$

52. $$\tan^2 \tfrac{1}{4} E = \tan \tfrac{1}{2} s \tan \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c).$$

The following diagram shows Prof. Blakslee's construction by which the direction ratios for plane right triangles give directly from a figure the analogies for a right trihedral or for a right spherical triangle:

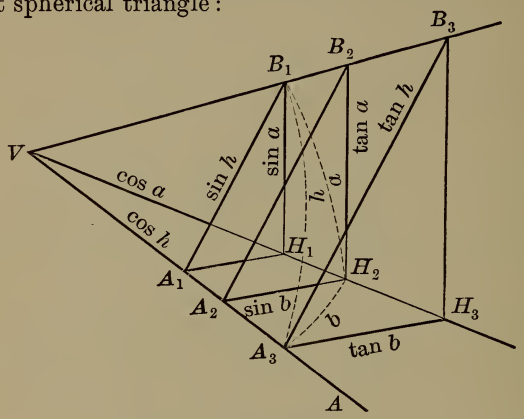


FIG. 123

The construction consists of two parts.

1. Lay off from the vertex V a unit's distance on each edge.

2. Pass through the three extremities of these distances three planes perpendicular to one of the edges, as VA. Now these three parallel planes will cut out three similar right triangles. The first being constructed in either of the two usual ways, the construction of the others is evident.

Since the plane angles A_1, A_2, A_3 all equal the dihedral A, and the nine right triangles in the three faces give the values in the figure, we have:

(1) $\sin A = \sin a : \sin h$; similarly, $\sin B = \sin b : \sin h$.

(2) $\cos A = \tan b : \tan h$; similarly, $\cos B = \tan a : \tan h$.

(3) $\tan A = \tan a : \sin b$; similarly, $\tan B = \tan b : \sin a$.

(4) $\cos h = \cos a \cos b$; by (3), $\cos h = \cot A \cot B$.

(5) $\sin A = \cos B : \cos b$; $\sin B = \cos A : \cos a$.

NOTE. If a sphere of unit radius is described about V as a centre, the three faces will cut out a right spherical triangle, having the sides a, b, and h, and angles A, B, and H. The above formulas are thus seen to be the analogies of:

(1) $\sin A = a : h$; $\sin B = b : h$.

(2) $\cos A = b : h$; $\cos B = a : h$.

(3) $\tan A = a : b$; $\tan B = b : a$.

(4) $h^2 = a^2 + b^2$; $1 = \sin^2 + \cos^2$; $1 = \cot A \cot B$.

(5) $\sin A = \cos B$; $\sin B = \cos A$.

Napier's Rules give only the following, which follow from the analogies as numbered:

By (1) $\left\{\begin{array}{l} \sin a = \sin A \sin h = \tan b \cot B \\ \sin b = \sin B \sin h = \tan a \cot A \end{array}\right\}$ (3)

(5) $\left\{\begin{array}{l} \cos A = \sin B \cos a = \tan b \cot h \\ \cos B = \sin A \cos b = \tan a \cot h \end{array}\right\}$ (2)

(4) $\{\cos h = \cos a \cos b = \cot A \cot B\}$ (4)

GAUSS'S EQUATIONS

$$\cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2} c = \cos \tfrac{1}{2}(a + b) \sin \tfrac{1}{2} C.$$

$$\sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2} c = \cos \tfrac{1}{2}(a - b) \cos \tfrac{1}{2} C.$$

$$\cos \tfrac{1}{2}(A - B) \sin \tfrac{1}{2} c = \sin \tfrac{1}{2}(a + b) \sin \tfrac{1}{2} C.$$

$$\sin \tfrac{1}{2}(A - B) \sin \tfrac{1}{2} c = \sin \tfrac{1}{2}(a - b) \cos \tfrac{1}{2} C.$$

SURVEYING

CHAPTER I

FIELD INSTRUMENTS

SECTION I

DEFINITIONS

Definition. Surveying is the art of determining and representing distances, areas, and the relative position of points on the surface of the earth.

Classification. Of surveying there are various kinds, depending upon the extent, the purpose, or the method of the survey. The following are the principal divisions:

1. *Plane Surveying*, in which the part of the earth's surface surveyed is regarded as a plane; *Geodetic Surveying*, in which the true figure of the earth is regarded.

2. *Land Surveying*, in which boundary lines, contents, and outline maps are the chief things aimed at; *Topographic Surveying*, in which differences in elevation and contour maps are chiefly sought; *Hydrographic Surveying*, in which the purpose is to determine the configuration and topography of the bed or basin of a body of water; *Mine Surveying*, in which the position and extent of underground excavations are determined and graphically represented.

3. *Rectangular Surveying*, in which a system of perpendicular lines is used as reference lines; *Triangular Surveying*, which proceeds by means of a system of triangles referred to a well established base line.

Operations Comprised. Surveying commonly comprises the following three distinct operations:

1. The *Field Measurements,* or the determining certain lines and angles by direct measurement.

2. The *Computation* of the required parts from the measured lines and angles.

3. The *Plotting,* or representing on paper the measured and the computed parts in relative extent and position.

Historic Note. Surveying is undoubtedly one of the oldest of the arts of civilized man. The Bible contains several admonitions not to remove "the ancient landmark," as in Proverbs xxii. 28. To the Babylonians is credited the division of the circle into 360 degrees. The Egyptians were known to survey frequently the valley of the Nile, a necessity owing to the periodic overflow of that river. Thence came *Geometry.* The Egyptians also possessed rules for finding the area of land of various shapes. Moreover, on Egyptian soil the Greek mathematician Eratosthenes made the first attempt at determining the circumference of the earth by measuring an arc of the circumference. This was in 276 B.C. Among the Romans Surveying was considered one of the liberal arts, and received impetus in the time of Julius Cæsar from his sweeping order that the entire empire should be surveyed for the purpose of equitable adjustment of taxes, and also from the introduction of the more practical parts of Greek Geometry. The works of Roman surveyors served as models for centuries, and much that we have to-day is only improvements on what has been handed down from them. For a brief account of surveying in the United States, see Cajori's "The Teaching and History of Mathematics in the United States," pp. 92, 286.

SECTION II

THE CHAIN

Surveyor's Chain. The Surveyor's Chain, or Gunter's Chain as it is often called, is made of iron or steel wire and is 4 rods or 66 feet long, composed of 100 links connected by small rings, and provided with a tally mark at the end of every 10 links. A link as a unit of measure includes a link of the chain and half the rings that connect it with adjoining links. Each link is 7.92 inches long. Since a chain is 4 rods long, a square chain contains 16 square rods, and since an acre contains 160 square rods, a square chain is one-tenth of an acre. A square chain contains also 10,000 square links and, therefore, an acre contains 100,000 square links. Hence, if a given area is expressed in square chains, it is reduced to acres by pointing off the last figure, and, if expressed in square links, it is reduced to acres by pointing off the last five figures. The tally marks are appropriately notched to facilitate counting links from either end, the one at the middle of the chain being rounded so as to be distinguished readily from the others. Handles form part of the end links, to which they are so attached as to prevent twisting and to allow lengthening or shortening of the chain. The Surveyor's Chain is used in measuring land.

Engineer's Chain. The Engineer's Chain differs from the ordinary Surveyor's Chain chiefly in that it is 100 feet in length, the length of each link being 1 foot. It is used in surveying railroads and canals, and often in other surveys where extensive lines are to be run.

Both the Surveyor's Chain and the Engineer's Chain are generally provided with attachments, so that from the full chains half-chains can be made up, to be used in case of rough or hilly country.

Accompanying Pieces. Usually eleven, sometimes ten, *Marking pins* go with the chain. These are of iron or steel, about 14 inches long, pointed at one end and formed into a ring at the other end. In case eleven pins are used, the first one is placed at the beginning of the line to be measured, and thereafter one at the end of each chain. The last pin in the ground is, therefore, not to be counted. In case ten pins are used, the first one is placed at the end of the first chain, and so on, the last pin in the ground being counted. Strips of red cloth should be fastened to the ring ends of the pins so as to make them easily visible. *Ranging poles*, which are of various lengths, are necessary for alignment. These are commonly made of wood, and are steel shod, graduated to feet, and painted in alternate red and white stripes.

How to chain. Ranging poles should be placed, one at each end of the line to be measured, and at such intermediate points as the necessities of the case require. A head chainman or leader, and a rear chainman or follower are required. The follower takes one end of the chain, and one pin, which he thrusts into the ground at the beginning of the line. The leader takes the other end of the chain and the remaining ten pins, and moves forward until the word "Halt" from the follower warns him that he has advanced nearly the length of the chain. At this signal he stops, and the follower, meanwhile having placed his end of the chain against the pin at the beginning of the line, directs the leader by the words "Right" and "Left" until he is exactly in the line. This being accomplished, and the chain tightly stretched in a horizontal position, the follower says, "Down." The leader then puts in a pin at the end of the chain and answers, "Down"; after which the follower withdraws the pin at his end of the chain, and the chainmen move forward, repeating the process just described until the end of the line is reached.

If the marking pins in the hands of the leader are all placed

before the end of the line is reached, after putting the last pin in the ground he waits until the follower comes up to him, gives him the ten pins in his hands and records the fact that ten chains have been measured. The measuring then proceeds as before. If the distance from the last pin to the end of the line is less than a chain, the leader places his end of the chain at the end of the line, and the follower stretches tightly such part of the chain as is necessary to reach the last pin, and the number of links is counted. If the ground slopes, one end of the chain must be raised until the horizontal position is attained. By means of a plumb line or a slender staff or, less accurately, in case of the leader by dropping a pin (heavy end downwards), the point vertically under the raised end of the chain may be determined. If the slope is considerable, half a chain or less may be used; in which case care must be taken that the correct number of full chains and links is found. For instance, if a tally shows 15 half chains and 35 links, the appropriate measure is 7 chains and 85 links, or, as it is usually expressed, 7.85 chains.

Special Constructions by Means of the Chain. 1. At a given point in a given line to construct a perpendicular to that line.

Let LE (Fig. 1) be the given line, and P the given point. On LE measure off $PB = PA = 20$ links. Then place one end of the chain at B and the other end at A. Stretch the chain from the middle point, and mark that point, as C. PC is the perpendicular required. (Why ?)

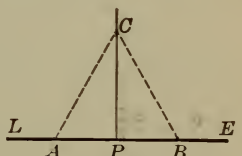


FIG. 1

Or, make $PB = 30$ links. Place one end of the chain at P, and the end of the 90th link at B. Then, taking the chain at the end of the 40th link from P and stretching both portions tightly, mark that point, as C. Then PC is the perpendicular required. (Why ?)

2. Through a given point without a given line to construct a perpendicular to that line.

Let LE (Fig. 1) be the given line, and C the given point. Take any point as B in the line and stretch the chain between C and B; then swing the chain about C until the point at B is again in the line, as at A. Measure the distance between A and B. Then P, the mid-point of AB, is a second point in the required perpendicular. (Why?)

Or, let the middle of the part of the chain between C and B be held in place, and swing the end at C until it meets the line as at P. PC is the required perpendicular. (Why?)

3. At a given point in a given line to construct an angle equal to a given angle.

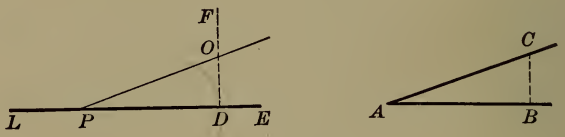


FIG. 2

Let P (Fig. 2) be the given point in the given line LE, and angle A the given angle. Make $PD = AB$. At D and B, respectively, construct perpendiculars DF and BC. Make $DO = BC$. Then angle OPD is the angle required. (Why?)

4. To construct any given angle, as 25° 40′.

Find from the tables the tangent of 25° 40′, which is 0.4806. Lay off PD (Fig. 2) = 100 links. Construct the perpendicular DF and lay off $DO = 48.06$ links. Then angle OPD is the angle required. (Why?)

5. Through a given point to construct a line parallel to a given line.

Let P (Fig. 3) represent the given point, and LE the given line. Through P lay out any convenient line as BA

intersecting LE. Construct angle BPD = angle PAE. Then the line CD is the required line. (Why?)

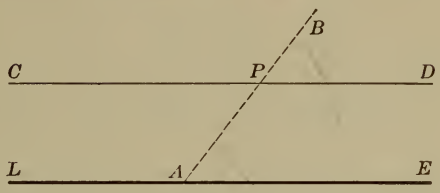


FIG. 3

Obstacles to chaining. In general practice various obstacles are encountered in chaining. The circumstances in each case must decide the best method to be used. Only a few suggestive cases can be considered in this work.

1. To measure a line when a building, or other object, stands in the way.

In Fig. 4 construct the perpendicular AB, the perpendicular BC, the perpendicular $CD = AB$, then the perpendicular DE, which will be in line LA prolonged.

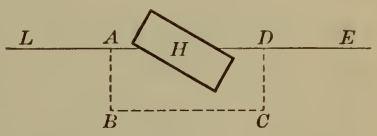


FIG. 4

Then, $LA + BC + DE = LE$. (Why?) As a check, another series of perpendiculars may be constructed.

2. To measure across a body of water.

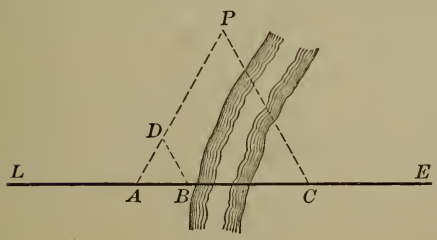


FIG. 5

At A (Fig. 5) lay out AP, making angle $PAB = 60°$. This can be done by laying out the equilateral triangle ABD. At P range out PC, making angle $APC = 60°$. Then measure AP.

The line AC is equal to AP. (Why?) If C is some fixed point in LE, across the stream, accessible or inaccessible, we may proceed as follows: After laying out AP, as already described, with 90 links of the chain stretched in the form of an equilateral triangle, and with one side of this triangle in AP, move the triangle until the point C is in line with the forward side of the triangle. Then proceed as before.

3. To measure a line the end of which is invisible from the beginning, and the intermediate points are unknown.

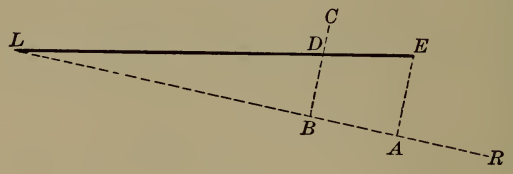


FIG. 6

Let LE (Fig. 6) represent the line. Lay out the line LR so that R shall be beyond E and visible from L. Construct from E the perpendicular EA to LR. Measure LA and AE. LE can then be computed. (How?) If intermediate points on LE are to be sought, take any point in LA, as B; construct BC perpendicular to LA; then measure off BD of such length that $BD : AE = LB : LA$. The line LR is called a *random line.*

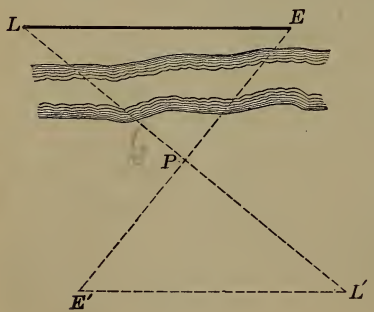


FIG. 7

4. To measure the distance between two inaccessible points.

Let L and E (Fig. 7) be two inaccessible points. Select some point as P from which both L and E are visible. Measure PL and PE by the method in 2. Range

out PL' in line with LP and equal to LP; similarly, $PE' = EP$. Then measure $L'E'$, which is equal to LE. (Why?)

EXERCISE I

1. Range out a line which, by estimation, is more than 10 chains long. Then measure it with the chain out and back.

2. Prolong a line beyond a building, or other obstacle which prevents continuous alignment.

3. Find the distance from a point to a line when the distance is more than a chain.

4. Lay out a square field each side of which shall be 5.76 chains long.

5. Find the length of a line by means of a random line. Then, as a check, find its length by direct measurement.

SECTION III

THE TAPE

Kinds of Tape. The tape measure used by the surveyor or engineer consists of a thin ribbon of steel, or of linen with interwoven wires of brass, wound upon a reel, often in a leather or metal case. Tapes vary in length from 25 feet to 500 feet or more. They are variously graduated to links, to feet and inches, to feet and tenths of a foot, to metric units, or to a combination of these. A common combination is feet and tenths of a foot on one side, and links on the reverse side.

Uses. The kind of tape determines to a great extent the use to which it is to be put. If 33 feet or 66 feet long and graduated to links, the evident purpose is for land surveying. If 50 feet or 100 feet long and graduated to feet and

tenths of a foot, the tape is especially designed for city work. Other kinds are employed in bridge, road, or mining work, in very accurate measurements of base lines, or as standards of comparison for other instruments of measurement.

SECTION IV

THE COMPASS

Parts and their Uses. The essentials of the compass, one style of which is shown in Fig. 8, are: the *compass circle*, graduated to half degrees and figured from 0° to 90° each way

FIG. 8. THE SURVEYOR'S COMPASS

NOTE. The letters E and W on the face of the compass are reversed from their true positions. The reason for this is that if the sights are turned towards the west, the north end of the needle is turned towards the letter W, and if the north end of the needle is turned towards E, the sights are turned towards the east.

If the north end of the needle points exactly towards E or W, the sights range east or west.

from the north and south points, for indicating the directions of lines; the *magnetic needle*, pivoted on a pin at the centre of the compass circle, for showing the direction of the magnetic meridian; and the *sight standards*, attached to the ends of the *main plate*, for alignment. To the main plate are attached two spirit levels at right angles to each other for leveling the instrument; underneath is a needle-lifting screw which, by actuating a concealed spring, lifts the needle from the pivot and presses it against the glass covering of the compass circle when the instrument is not in use; a tangent screw, and almost directly under it a clamp screw, which operates the *vernier;* and a small dial plate for keeping tally in chaining. The north end of the needle usually has some ornamentation to distinguish it from the south end, and a coil of fine wire is wound on the south end to prevent the needle from dipping. The sight standards have fine slits nearly their whole length, with circular openings at intervals to facilitate sighting upon an object; on the edges of the north standard are *tangent scales* for reading vertical angles, and on the outside of the south standard are two *eyepieces* at the same distance from the main plate as the zeros of the tangent scales, respectively. The *telescopic sight* is an attachment to the south standard, now often used. The instrument entire turns horizontally upon the upper end of a *ball spindle*, the lower end of which rests in a spherical socket in the top of a *Jacob's staff*, or a *tripod*, which supports the instrument. The *socket* of the compass which fits to the ball spindle is provided with a clamp screw and a spring catch. From the centre of the plate at the top of the tripod a plummet is suspended by which the centre of the compass can be placed directly over a definite point on the ground.

Kinds of Compasses. The compass described is the *vernier compass*, or *surveyor's compass*, and is the one in general use. If there is no vernier attachment, the compass is called a *plain*

compass and is used in running new lines and the preparation of maps. A *railroad compass* has all the features of the vernier compass, and has also a vernier plate and graduated limb for measuring horizontal angles.

Hints on the Use and Care of Instruments. The instruments described in this work are adjusted by the maker. If they should require readjustment, full directions will be found in the manual furnished with the instruments. Before beginning to use any instrument, make a thorough study of its various parts and their uses. In moving or adjusting any part always know what you are doing and why you are doing it. When an instrument is not in use keep it in a place that is free from moisture and dust.

Bearing of a Line. The *magnetic meridian* of a place is the direction which a bar magnet assumes when freely supported in a horizontal position. The *magnetic bearing* of a line is the angle it makes with the magnetic meridian. To take the bearing of a line proceed as follows: Place the compass so that the Jacob's staff, or plummet of the tripod, is directly over one end of the line, and level by pressing with the hands on the main plate until the bubbles are brought to the centres of the spirit levels. Turn the south end of the instrument toward you, and sight at the ranging pole at the other end of the line. Read the bearing from the north end of the needle. First, write N. or S. according as the north end of the needle is nearer N. or S.

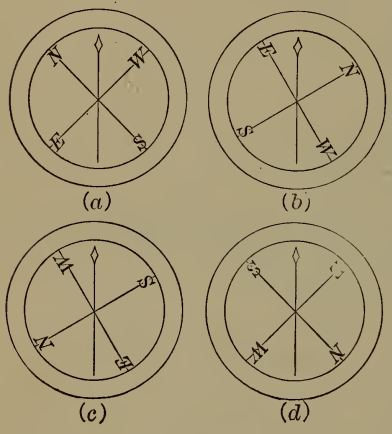


FIG. 9

of the compass circle; secondly, write the number of degrees between the north end of the needle and the nearest zero mark; thirdly, write E. or W. according as the north end of the needle is nearer E. or W. of the compass circle. Thus, in Fig. 9 (*a*), the bearing is N. 45° W.; (*b*), N. 60° E.; (*c*), S. 60° W.; (*d*), S. 45° E.

If the needle coincides with the N.S. or E.W. line, the bearing is N., S., E., or W. according as the north end of the needle is over N., S., E., or W. As the compass circle is divided into half degrees, the bearing may be determined pretty accurately to quarter degrees.

It will be noticed that the letters E and W on the face of the compass are reversed from their true positions. These are so placed in order that when the sights are turned towards the west the north end of the needle will point towards the letter W, or if the sights are turned towards the east, the north end of the needle will point towards the letter E. It turns out that if the south sight standard is always turned towards the observer, the reading at the north end of the needle will indicate the true bearing of the line. Should the north sight standard be turned towards the observer, the reading at the south end of the needle would then be taken.

Checking Bearings. When the bearing of a line has been taken, the instrument should be removed to the other end of the line and the reverse bearing taken. The number of degrees should be the same, but the letters should be reversed. For instance, if the direct bearing is N. $35\frac{3}{4}°$ W., the reverse bearing should be S. $35\frac{3}{4}°$ E. In case the reverse bearing is not what it ought to be, there is some mistake, or some local disturbance, or both. To detect errors a second trial at the direct bearing should be taken. To detect local disturbances take the direct and reverse bearings of other lines ranged out from the beginning of the line whose bearing is sought. If they all show the same difference between their two respective bearings, the evidence of some local disturbance, as iron,

iron ore, etc., is pretty conclusive. In this case the true bearing of the line can be obtained by making the necessary correction. In all cases, precautions should be taken to have the chain, pins, and other instruments that would affect the direction of the needle sufficiently removed from the compass.

Obstacles. When a fence or other obstruction interferes with placing the instrument over the line the instrument may be placed at one side, the ranging pole being correspondingly placed at the other end. If one end of the line cannot be seen from the other end, run a random line. Then (Fig. 6, p. 216) $\tan ELA = AE \div LA$, whence the angle ELA can be found. This angle combined with the bearing of the random line will give the bearing required. Or some point can be selected from which the ends of the line are visible. The distances to the ends may be measured, and the angle between the two auxiliary lines may also be measured. Of the triangle thus formed, the angle at the beginning of the given line may be computed, and, when properly combined with the bearing of the first auxiliary line, will give the required bearing. If a single triangle is not sufficient, a series of triangles may be employed until the end of the line is reached.

Measurement of Horizontal Angles. To measure a horizontal angle by means of the needle, place the compass over the vertex of the angle, take the bearing of each line separately, and combine these bearings according to the following rules, as suggested by Fig. 10:

1. If the first letters of the bearings are alike, and also the last letters, find the difference of the bearings.

2. If the first letters are alike, and the last letters unlike, add the bearings.

3. If the first letters are unlike, and the last also unlike, subtract the difference of the bearings from 180°.

4. If the first letters are unlike, and the last alike, subtract the sum of the bearings from 180°.

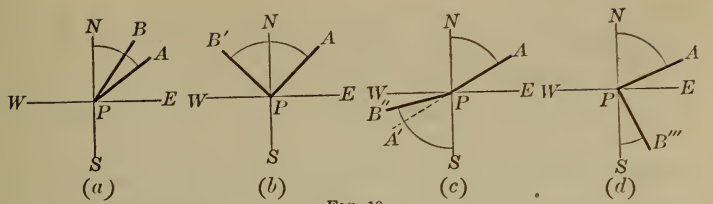


FIG. 10

Measurement of Vertical Angles. A *vertical angle* is an angle the sides of which are in a vertical plane. If one side of a vertical angle is horizontal and the other ascends, the angle is called an *angle of elevation;* if one side is horizontal and the other descends, the angle is called an *angle of depression. To measure an angle of elevation* by means of the compass, sight through the lower eyepiece to a point that is as far above the point whose elevation is sought as the instrument is above the point from which the elevation is to be taken. Read off the degrees of the right-hand tangent scale, marked by a card placed squarely across the face of the south standard, the top of the card being in the line of sight. *To measure an angle of depression*, proceed in the same manner, using the upper eyepiece and the left-hand tangent scale. If the compass is provided with a telescopic sight that has a vertical circle attachment, these should be used instead of the eyepieces and tangent scales.

Verniers. A vernier is a contrivance for measuring portions smaller than those into which a line is divided. We shall describe two kinds.

Let AB (Fig. 11) be a portion of a line graduated to tenths and hundredths of a foot. VR is the vernier.

In (a), nine parts of the line are divided into ten equal parts on the vernier. Hence, a division on the vernier is less than a division on the line by the difference between $\frac{1}{10}$ of a foot and $\frac{1}{10}$ of $\frac{9}{100}$ of a foot, or $\frac{1}{100}$ of a foot. Now, if the vernier

is moved so that 1 of the vernier coincides with 1 of the scale, it has moved over a space equal to $\frac{1}{1000}$ of a foot. If the vernier is moved so that 2 of the vernier coincides with 2 of the scale, it has moved over a space equal to $\frac{2}{1000}$ of a foot; and so on.

In (*b*), 6 of the vernier coincides with 9 of the scale, which indicates that the zero of the vernier has moved past 3 of the scale a space equal to $\frac{6}{1000}$ of a foot. The reading, then, is

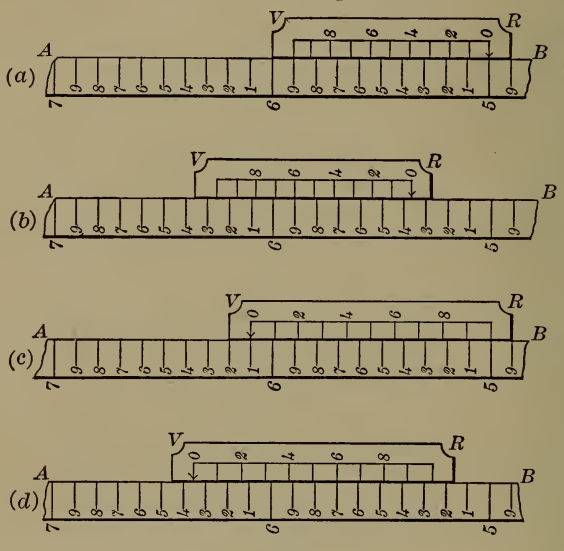


FIG. 11

0.536 foot. This form of the vernier is known as the direct form, since the figuring on the vernier proceeds in the same direction as that on the scale.

In (*c*), eleven parts of the line are divided into ten equal parts on the vernier. Hence, a division on the vernier is greater than a division on the line by the difference between $\frac{1}{10}$ of $\frac{1}{100}$ of a foot and $\frac{1}{100}$ of a foot, or $\frac{1}{1000}$ of a foot. Now, if the vernier is moved so that 1 of the vernier coincides with 10

of the scale, *i.e.*, the end of the 6th tenth, the vernier has moved over a space equal to $\frac{1}{1000}$ of a foot. If the vernier is so moved that 2 of the vernier coincides with 9 of the scale, the vernier has moved over a space equal to $\frac{2}{1000}$ of a foot; and so on.

In (*d*), 6 of the vernier coincides with 7 of the scale, which indicates that the zero of the vernier has moved past 3 of the scale a space equal to $\frac{6}{1000}$ of a foot. The reading here is 0.636 foot.

This form of the vernier is known as the retrograde form, since the figuring on the vernier proceeds in the opposite direction from that on the scale. In either form the following rule for using and reading the vernier may be adopted:

Move the vernier until its zero line, or index, is at the point to which the required measurement is to be taken; read the main scale to the nearest division below the index, and that number of the division line of the vernier which stands opposite a line of the main scale.

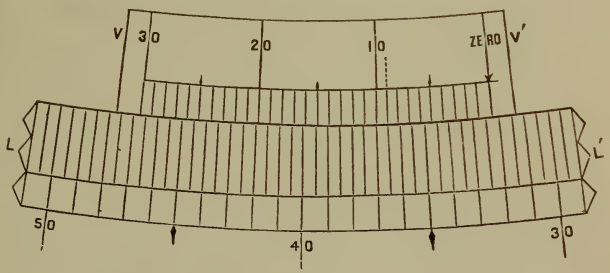


FIG. 12

Compass Vernier and its Uses. Let LL' (Fig. 12) represent the limb of the compass graduated to half degrees, and VV' the vernier divided into thirty equal spaces, equal to twenty-nine spaces of the limb. Then, one space of the vernier is less than one space of the limb by $1'(=30'-\frac{1}{30}$ of $29\times30')$, and the reading may be obtained to single minutes.

In Fig. 12 the index, or zero, of the vernier stands between 32° and 32° 30′, and the line of the vernier marked 9 coincides with a line of the limb. Hence, the reading is 32° 9′.

When the index moves from the zero line of the limb in a direction opposite to that in which run the numbers of the limb, the number of minutes obtained as above must be subtracted from 30′ to obtain the minutes required. (Why ?) If, however, the vernier is made double, that is, if it has thirty spaces on each side of the zero line, it is always read directly. The usual form of the double vernier, shown in Fig. 13, has

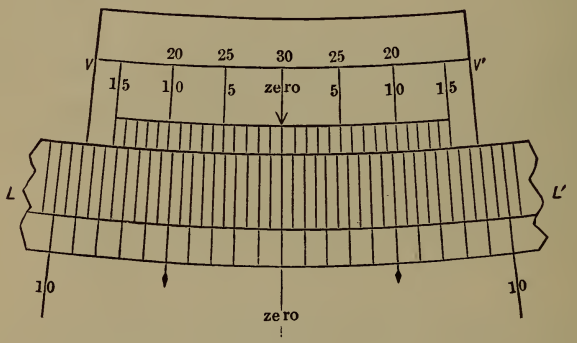


FIG. 13

only fifteen spaces on each side of the zero line. When the vernier is turned to the *right* less than 15′ past a division line of the limb, read the lower figures on the *left* of the zero line at any coincidence; if moved more than 15′ past a division line of the limb, read the upper figures on the *right* of the zero line at any coincidence; and *vice versa*. In this form of the double vernier it will be observed that the spaces on the vernier are larger than those on the limb, since the 30 equal spaces of the former are equal to 31 half-degree spaces of the latter.

The most important use of the vernier compass is in setting off the variation of the needle explained just below. If the variation of the needle at any place is known, by means of the vernier screw the compass circle may be turned through an arc equal to the variation. If the observer stands at the south end of the instrument, the vernier is turned to the right or left according as the variation is west or east. The compass now gives the bearings of the lines with the *true meridian.*

In order to retrace the lines of an old survey, turn the sights in the direction of a known line and move the vernier until the needle indicates the old bearing. If no line is definitely known, the *change of variation* from the time of the old survey will give the arc to be set off.

Magnetic Declination. The *magnetic declination*, or *variation of the needle*, at any place is the angle which the magnetic meridian makes with the true meridian, or north and south line. The variation is *east or west*, according as the north end of the needle lies east or west of the true meridian. Western variation is indicated by the sign +, and eastern by the sign −. The kinds of magnetic declination are put under three heads:

1. *Irregular variations*, which are sudden deflections of the needle due to magnetic storms or other causes not well understood.

2. *Solar-diurnal variations*, which in northern latitudes reach their farthest point east about 8 o'clock A.M., and their farthest point west about 2 o'clock P.M., varying from 5′ in the winter in some localities to 20′ in the summer in other localities.

3. *Secular variation*, which is a change in the same direction for a period of years, then in the opposite direction for about the same time.

It is not accurately known how long it takes a complete secular variation to run its course, but from data already obtained it seems probable that the period of time covered is not less than two and a half or three centuries.

Place	Latitude		Longitude		Variation							Annual Change
					1800	1820	1840	1860	1880	1890	1900	1900
	Deg.	Min.	Deg.	Min.	Degrees	Degrees	Degrees	Degrees	Degrees	Degrees	Degrees	Minutes
Halifax, N.S.	44	40	63	35	15.9	17.4	18.9	19.9	20.6	20.7	20.7	−0.3
Eastport, Me.	44	54	66	59	13.2	14.8	16.4	17.79	18.71	18.92	19.0	0.0
Bangor, Me.	44	48	68	47	10.9	12.1	13.7	15.3	16.54	16.99	17.1	+1.2
Provincetown, Mass.	42	03	70	11	7.2	8.2	9.61	11.00	12.12	12.51	12.9	1.3
Portland, Me.	43	39	70	17	8.50	9.46	10.82	12.29	13.58	14.08	14.4	2.0
Portsmouth, N.H.	43	04	70	42	7.6	8.3	9.55	11.03	12.40	12.94	....	2.2
Boston, Mass.	42	22	71	04	6.90	7.78	9.01	10.33	11.47	11.9	12.3	1.4
Cambridge, Mass.	42	23	71	08	7.10	7.97	9.29	10.63	11.59	11.9	12.4	1.6
Quebec, Canada	46	48	71	14	12.1	12.3	13.8	16.0	17.4	17.5	17.5	−0.7
Providence, R.I.	41	50	71	24	6.46	6.71	8.24	9.78	10.79	11.56	12.0	+2.5
Hartford, Conn.	41	46	72	40	5.10	5.58	6.59	7.93	9.29	9.89	10.4	2.8
New Haven, Conn.	41	18	72	55	4.7	5.0	5.95	7.35	8.84	9.52	9.9	3.1
Burlington, Vt.	44	28	73	12	7.2	7.78	8.90	10.27	11.58	12.11	12.5	2.2
Williamstown, Mass.	42	43	73	13	5.7	6.3	7.4	8.8	10.3	10.9	11.4	2.8
Montreal, Canada	45	30	73	35	8.0	7.9	9.4	12.0	13.8	14.4	15.4	4.2
Albany, N.Y.	42	40	73	45	5.5	6.02	7.07	8.44	9.87	10.52	11.1	3.2
New York, N.Y.	40	43	74	00	4.3	4.61	5.61	6.91	7.90	8.49	9.1	3.6
New Brunswick, N.J.	40	30	74	27	2.54	3.43	4.66	5.98	7.12	7.55	7.9	1.6
Cape Henlopen, Del.	38	47	75	05	0.8	1.1	2.00	3.36	4.86	5.6	6.2	3.5
Philadelphia, Pa.	39	58	75	10	2.1	2.44	3.46	4.73	6.20	6.97	7.7	3.8
Cape Henry, Va.	36	56	76	00	0.24	0.25	0.82	1.80	2.94	3.5	4.0	2.7
Ithaca, N.Y.	42	27	76	29	3.3	2.7	3.1	4.1	5.71	6.58	7.5	5.1
Baltimore, Md.	39	16	76	35	0.64	0.88	1.70	2.90	4.17	4.74	5.4	2.8
Williamsburg, Va.	37	16	76	43	−0.17	−0.22	0.38	1.47	2.75	3.3	3.9	....
Harrisburg, Pa.	40	16	76	53	0.0	0.8	2.2	3.71	5.05	5.52	6.0	1.7
Washington, D.C.	38	53	77	00	−0.1	0.3	1.19	2.42	3.66	4.18	4.7	2.5
Newbern, N.C.	35	07	77	03	−2.1	−1.66	−0.70	0.54	1.74	2.25	2.6	2.1
Buffalo, N.Y.	42	55	78	54	0.22	0.41	1.35	2.84	4.51	5.30	6.0	4.0
Toronto, Canada	43	39	79	23	....	....	1.32	2.17	3.62	4.12	....	4.0

Place	Latitude		Longitude		Variation							Annual Change
					1800	1820	1840	1860	1880	1890	1900	1900
	Deg.	Min.	Deg.	Min.	Degrees	Degrees	Degrees	Degrees	Degrees	Degrees	Degrees	Minutes
Charleston, S.C.	32	47	79	56	−4.55	−4.05	−3.03	−1.73	−0.45	0.09	0.5	+2.3
Pittsburg, Pa.	40	28	80	01	....	....	0.18	1.26	2.49	3.06	3.6	2.9
Erie, Pa.	42	09	80	05	−0.46	−0.39	0.36	1.60	2.99	3.62	4.2	3.0
Savannah, Ga.	32	05	81	05	....	−4.7	−4.2	−3.27	−2.06	−1.45	−0.8	3.6
Cleveland, O.	41	30	81	42	−1.8	−1.4	−0.66	0.39	1.52	2.05	2.5	2.7
Key West, Fla.	24	33	81	48	....	−6.86	−6.03	−4.85	−3.57	−2.96	−2.4	3.0
Detroit, Mich.	42	21	83	03	−3.1	−2.84	−2.04	−0.93	0.23	0.74	1.2	2.3
Sault Ste. Marie, Mich.	46	30	84	20	−0.5	−1.1	−1.04	−0.34	0.84	1.52	2.2	4.0
Cincinnati, O.	39	08	84	25	−4.89	−4.99	−4.51	−3.57	−2.39	−1.80	−1.3	3.3
Grand Haven, Mich.	43	05	86	13	....	−5.0	−5.2	−4.45	−2.73	−1.5	−1.0	....
Nashville, Tenn.	36	09	86	48	....	−6.7	−6.9	−6.3	−5.13	−4.40	3.6	4.7
Michigan City, Ind.	41	43	86	54	....	....	−5.4	−4.6	−3.5	−2.9	−1.8	4.5
Pensacola, Fla.	30	21	87	16	−6.84	−7.50	−7.43	−6.65	−5.34	−4.59	−4.1	4.3
Chicago, Ill.	41	50	87	37	....	−6.1	−6.2	−5.7	−4.52	−3.81	−3.1	4.4
Milwaukee, Wis.	43	04	87	53	....	....	....	−6.9	−5.4	−4.5	−3.6	5.4
Mobile, Ala.	30	42	88	03	−5.81	−6.71	−7.07	−6.75	−5.84	−5.23	−4.6	4.3
New Orleans, La.	29	59	90	05	−7.12	−7.96	−8.16	−7.66	−6.59	−5.91	−5.2	4.3
St. Louis, Mo.	38	38	90	16	....	....	−8.6	−7.7	−6.4	−5.6	−5.0	4.2
Duluth, Minn.	46	46	92	04	....	....	....	−10.02	−10.06	−9.9	−12.3	2.1
Galveston, Tex.	29	18	94	47	....	....	....	−8.84	−8.16	−7.56	−6.9	4.0
Omaha, Neb.	41	16	95	56	....	−12.6	−12.33	−11.47	−10.23	−9.56	−8.9	4.1
Austin, Tex.	30	17	97	44	....	....	−10.7	−9.74	−8.80	−8.34	−7.9	2.8
San Antonio, Tex.	29	27	98	28	....	−9.8	−10.29	−10.16	−9.43	−8.89	−8.3	3.8
Denver, Col.	39	45	105	00	....	....	....	−15.14	−14.52	−14.06	−13.5	3.5
Salt Lake City, Utah	40	46	111	54	....	....	....	−16.45	−16.58	−16.3	−16.1	2.1
San Diego, Cal.	32	42	117	14	−10.69	−11.79	−12.67	−13.21	−13.32	−13.22	−13.0	1.3
Seattle, Wash.	47	36	122	20	....	....	....	−21.8	−22.28	−22.25	−22.5	0.0
San Francisco, Cal.	37	48	122	27	−13.6	−14.54	−15.42	−16.10	−16.51	−16.58	−16.7	0.1
C. Mendocino, Cal.	40	26	124	24	−15.1	−16.0	−16.9	−17.4	−17.69	−17.69	−17.17	0.6

The *agonic line*, or line of no variation, is a line joining those places at which the magnetic meridian coincides with the true meridian. At the beginning of the present century this line crossed the United States in an irregular way from Michigan to South Carolina. It is gradually moving westward, so that the variation is increasing at places east of this line, and decreasing at places west of the line. East of this line the variation is westerly, and west of this line the variation is easterly. Lines that join places of equal magnetic declination are called *isogonic lines*.

Table of Magnetic Declination. On pp. 228, 229 will be found a table showing the variation in magnetic declination at different places in the United States and contiguous territory during the nineteenth century; also the annual change for the epoch of 1900.

EXERCISE II

1. Lay out a field of five sides and take the bearings and measures of the sides in order, beginning at the most westerly point and going about the field clockwise.

2. From the bearings obtained in Example 1 find the value of each of the interior angles. What is their sum?

3. Lay out the field the bearings and distances of whose sides are given in Example 1 of Exercise VI, p. 272.

4. Range out a line whose bearing is N. 38° 30′ W., and at some point in this line range out another line making a right angle with it. What is the bearing of the second line?

5. Set up the compass at a spot near which there is known to be some local disturbance, as iron in a building, or an iron fence, and find the variation of the needle due to such disturbance.

SECTION V

THE TRANSIT

Surveyor's Transit. The transit is the most important instrument used in surveying. There are many modifications of it, each adapted to its own particular use. All, however, have about the same essential features. The one described here, and shown in Fig. 14, is the *surveyor's transit*, the one of most general use. The essential parts are the *telescope* with its axis and two standards, the *circular plates* with their attachments, the *sockets* upon which the plates revolve, the *leveling head*, and the *tripod*. Within the telescope are two fine *cross wires*, at right angles to each other, whose intersection determines the *optical axis*, or *line of collimation*, of the telescope. Under the telescope, and attached to it, is a *spirit level* by which horizontal lines may be run, or the difference of level between two stations be found. The axis of the telescope carries a *vertical circle* which measures vertical angles to single minutes by means of a vernier. The *vernier plate*, which carries the telescope and also the compass circle, has two *verniers* diametrically opposite to each other, and it moves entirely around the *graduated limb* of the *main plate*. The sockets are compound; the interior spindle attached to the vernier plate turning in the exterior socket, when an angle is taken on the limb, but when the plates are clamped together the exterior socket itself, and with it the whole instrument, revolves in the socket of the leveling head. The transit is leveled by four *leveling screws* which pass through a plate firmly fastened to the ball spindle and rest in small sockets, these resting in turn on the upper side of the tripod plate. On the underside of this lower or tripod plate is an arrangement called a *shifting centre*, which enables the surveyor to change the position of the vertical axis horizontally without

moving the tripod; besides this there is, if specially ordered, a device called a *quick-leveling attachment* to bring the transit quickly to an approximately level position by the pressure of the hands after which the leveling screws are used.

Uses. The transit may be used for all the purposes for which the compass may be used, but with much greater precision. The principal use, however, is in measuring horizontal angles by means of the graduated limb and verniers. It may be used, furthermore, in obtaining differences of level; also, provided there is the attachment to the telescope known as the *stadia*, in measuring distances, especially over broken ground. A still further use, when the transit is supplied with what is known as a *gradienter attachment*, is in fixing grades as well as measuring distances.

Getting the Transit Ready. The instrument should be set up so as to be firm, the tripod legs being pressed into the ground until the plates are as nearly level as can conveniently be done by this means. For the subsequent leveling turn the instrument until the spirit levels on the vernier plate are parallel to the vertical planes passing through opposite pairs of the leveling screws. Take hold of opposite screw heads with the thumb and forefinger of each hand, and turn both thumbs in or out as is necessary to bring the bubble to its proper place, the left thumb always moving in the direction that the bubble is to move. For precise work, in addition to leveling by the leveling screws, it is advisable to level the plates by the telescope level, as this is much more sensitive than the levels on the plate. In this operation the position of the level on the telescope must be observed over both sets of leveling screws, one half the correction being made by the axis tangent screw, the other half by the leveling screws. Before an observation is made with the telescope, the eyepiece should be focused by its pinion until the cross wires appear distinct; the object glass is then focused by its pinion

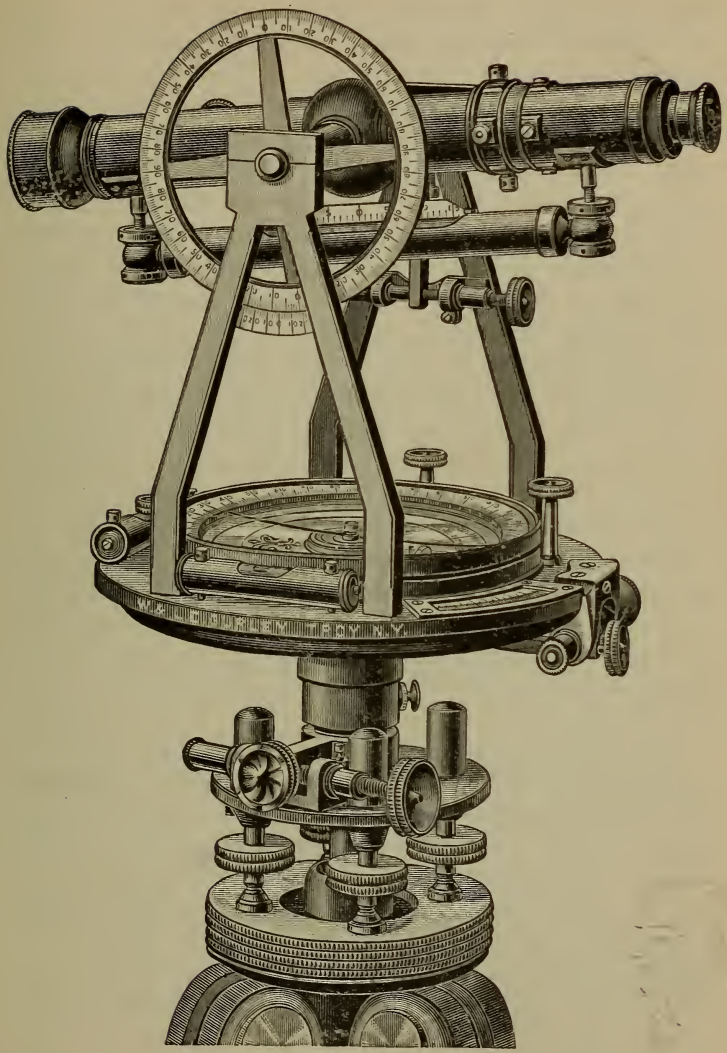


FIG. 14. THE SURVEYOR'S TRANSIT

until the object to be observed appears well defined. This latter process must be repeated when the distance to the object is changed.

Measurement of Horizontal Angles. Place the instrument directly over the vertex of the angle, and level. Set the limb at zero by the tangent screw of the plates, and turn the telescope in the direction of one of the sides of the angle, directing it to the object by the tangent screw of the leveling head. Then unclamp the main plate and turn the telescope until it is in the direction of the other side of the angle, and read the angle by the verniers. The object of the two verniers on the vernier plate is to correct any mistakes that might arise from the want either of exact coincidence in the centres of the verniers and the limb or of exact graduations on the limb. The correct reading may be obtained by adding to the reading of one vernier the supplement of the reading of the other, and taking half their sum.

Measurement of Vertical Angles. Direct the telescope to the object; clamp, and read the angle indicated on the vertical circle by the vernier. The angle read will be an angle of elevation or depression as the case may be, the horizontal line being the line of collimation of the telescope when in a horizontal position.

Stadia Measurements. As already stated on page 232, the stadia is an attachment to the telescope used in measuring distances, especially over rough ground. It consists essentially of two horizontal wires fastened to small movable slides, and so adjusted as to include a given space, say one foot on a rod 100 feet distant. These wires will then include two feet on a rod 200 feet away, or a half-foot at a distance of 50 feet, and so on. Usually the instrument is so adjusted that the zero of the indicated distance is in front of the centre of the instrument; hence, the true distance is the indicated distance plus the distance of this zero from

the centre of the instrument. This latter distance is determined for each instrument by the maker, and noted on a card placed on the inside of the instrument box. It is known as the *constant* of the instrument. The readings are taken on a rod, specially designed for the purpose, known as the *stadia rod*. This is graduated to feet, and tenths and hundredths of a foot. Any ordinary leveling rod, if similarly graduated, will answer the same purpose. Obviously in taking stadia measurements the rod must always be held at right angles to the line of sight. This statement has special reference to measurements taken up or down hill-slopes. In this case, if horizontal distance is required, the measured distance must be multiplied by the cosine of the angle of elevation or depression. (Why ?)

EXERCISE III

1. By means of the transit, measure the interior angles of the field of Example 1, Exercise II, p. 230, and compare with the results obtained in Example 2 of the same exercise.

2. Lay out the entire angular magnitude about some point into four or more angles, and measure each one of them. What should the sum of them equal ?

3. If the constant of a transit adjusted to one foot 100 feet away is 3.8 inches, what is the true length of a line when the indication on the rod is 2.35 feet ?

4. Measure a line by the stadia, and compare with measurements taken by the chain and also by the tape.

5. Compute the height of a tall object, as a tree or steeple, by first measuring its distance from some convenient point and measuring the angle of elevation at that point.

6. Lay out a square field containing just one acre.

SECTION VI

THE SOLAR COMPASS

Description and Uses. A full description of the *solar compass*, or *Burt's solar compass*, as it is often called from its inventor, with its principles, adjustments and uses, forms the subject of a considerable volume, which should be in the hands of the surveyor who uses this instrument. The limits of our space will allow only a brief reference to its principal features. Fig. 15 exhibits the instrument by itself; Fig. 16, p. 239, is a graphical illustration of the solar apparatus as an attachment to the transit, the circles shown being intended to represent those supposed to be drawn upon the concave surface of the heavens. The form of the solar compass shown in Fig. 15 has the arrangement of its sockets and plates similar to that of the transit, the standards similar to those of the compass, the solar apparatus being placed on the upper vernier plate and taking the place of the needle, for which it operates as a substitute in the field.

The solar compass consists mainly of three arcs of circles, *a* the *latitude arc*, by which is set off the latitude of the place, *b* the *declination arc*, by which is set off the declination of the sun, and *c* the *hour arc*, by which is set off the hour of the day. The arm *h* is movable about a point at the extremity of the piece containing the declination arc, there being at each end a *solar lens* having its focus on a silvered plate on the other end. The arc of the declination limb turns on an axis, and one or the other solar lens is used, according as the sun is north or south of the equator. Fig. 15 shows the position of the declination arc when the sun is south; Fig. 16, when it is north. The needle box is moved about its centre by a slow-motion screw. It contains a magnetic needle, and is furnished with a graduated arc about 36° in extent.

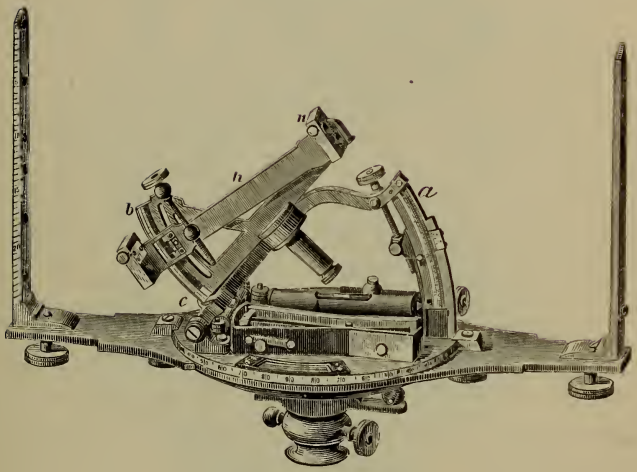


FIG. 15. BURT'S SOLAR COMPASS

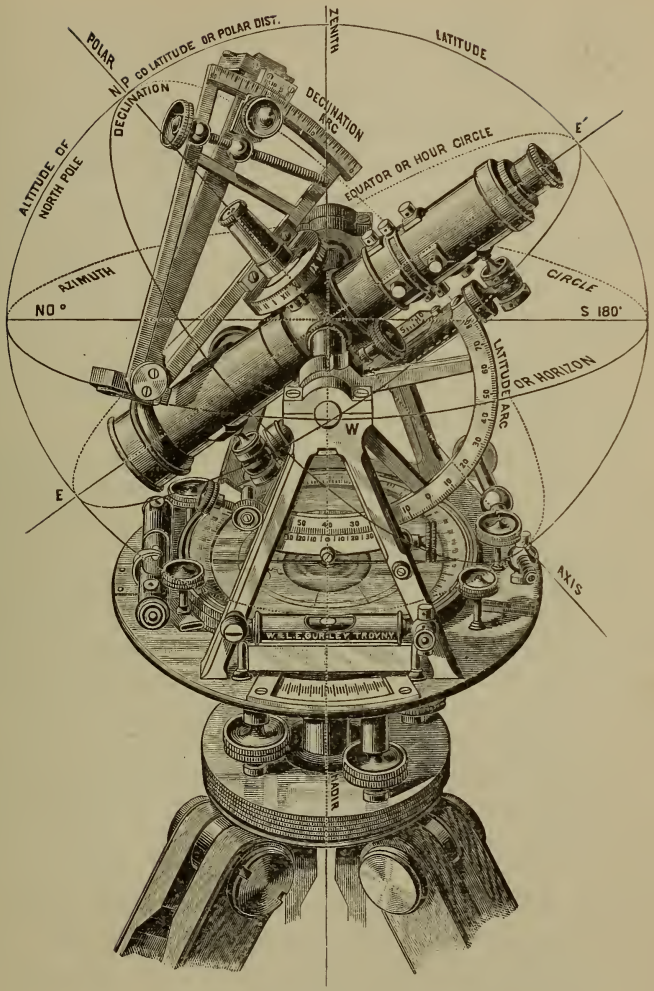


FIG. 16. TRANSIT WITH SOLAR ATTACHMENT

The circles shown in the cut are intended to represent in miniature circles supposed to be drawn upon the concave surface of the heavens.

The solar compass may be used for most of the purposes of a compass or transit. Its most important use, however, is to run north and south lines, especially in laying out the public lands. It may be used also in determining the latitude of a place.

To establish a True Meridian. Set off on the latitude arc the latitude of the place. Set off on the declination arc the declination of the sun, corrected for refraction. Set the instrument over the station; level, and turn the sights in a north and south direction by the needle. The surveyor then turns the solar lens to the sun, and with one hand on the instrument and the other on the revolving arm, moves both from side to side until the sun's image is made to appear on the silvered plate, precisely between the equatorial lines. The line of sights then indicates the true meridian.

The bearing of any line from the meridian may be read by the verniers of the horizontal limb. When a due east and west line is to be run, these verniers are set at 90°, and the sun's image is kept between the lines as before.

Other Methods. *By North Star at Culmination.* The North Star, or Polaris, at present revolves about the north pole of the heavens at the distance of about $1\frac{1}{5}°$; hence, it is on the meridian twice in 23 h. 56 m. 4 s. (a sidereal day), once above the pole, called the *upper culmination*, and 11 h. 58 m. 2 s. later below the pole, called the *lower culmination.*

The time of the upper culmination of Polaris may be found by means of the star Mizar, the middle one of the three stars in the handle of the Dipper (in the constellation of the Great Bear). It crosses the meridian at nearly the same time as Polaris. Suspend a plumb line, placing the bob in a pail of water to lessen its vibrations. South of the plumb line, upon a horizontal board firmly supported, place a compass sight, or any upright with a small opening or slit, fastened to a board a few inches square. At night, when Mizar by estimation approaches the meridian, place the compass sight in line with

Polaris and the plumb line, and move it so as to keep it in this line until the plumb line falls also on Mizar (Fig. 17). Note the time; then (1903) 3 m. 39 s. later Polaris will be on the meridian. If then Polaris, the plumb line and the compass sight are brought into line, the plumb line and compass sight will give two points in the meridian; or if the telescope of the transit is brought to bear on Polaris, and a light is held near to make the wires visible if necessary, the telescope will then lie in the plane of the meridian, which may be marked by bringing the telescope to a horizontal position.

FIG. 17

For each year subsequent to 1903 add 21 s. to 3 m. 39 s. If the lower culmination takes place at night, the time may be found in a similar manner. When Mizar cannot conveniently be used, δ Cassiopeiae (Fig. 17) may be employed, the method being the same as in the case of Mizar. The interval, however (1903), is 4 m. 24 s. and the annual increase of the interval about 20 s.

By North Star at Greatest Elongation. When Polaris is at its greatest apparent angular distance east or west of the pole, it is said to be at *greatest elongation.* It attains its greatest eastern elongation and western elongation, respectively, 5 h. 59 m. 1 s. after lower and upper culmination. The *azimuth* of a star is the angle which the meridian plane makes with the vertical circle passing through the star and the zenith of the observer.

If now we know the time of either extreme elongation and also the azimuth of Polaris at an extreme elongation, we can from these data establish a true meridian. The latter of these

data is given for various latitudes and for years to come in tables, to which the surveyor is supposed to have access. To obtain a line in the direction of Polaris at greatest elongation, we may proceed as follows: A few minutes before the time of greatest elongation, place the compass sight in line with the plumb line and Polaris, keeping it in line with these until the star begins to recede. At this moment the sight and plumb line are in the required line. Or bring the telescope of the transit to bear on the star, and follow it keeping the vertical wire over the star until it begins to recede. The telescope will then be in the required line. In either case, after having the transit sighted in the direction of the line just found, turn it in the proper direction through an angle equal to the azimuth as found from the tables.

The accompanying table * gives the Washington mean time of each *tenth* transit of Polaris (upper culmination) at the meridian of Washington, D.C. The last column contains the variation per day, to facilitate the interpolation of the time for any intermediate transit.

The transit which occurs October 17 is the *tenth* transit following that which occurs on October 8. This is because *two* transits occur on October 13; the interval separating them being 23 h. 56 m. 4 s. of mean time. These two transits are introduced in the table for greater convenience, and as a safeguard against error respecting the particular day of transits in that vicinity. The double lines merely call attention to the break thus caused in the series.

By interpolation we may, by taking account of the *longitude* of any given station, find the local mean time of transit of Polaris at that station for any particular day. Thus, to find the Cincinnati mean time of the upper culmination of Polaris at Cincinnati, on May 15, 1902, we have (p. 244):

* Furnished by the Director of the Nautical Almanac Office, Washington, D.C.

Day of the Year	Local Mean Time of Every 10th Transit	Change in 1 Day
1902	(Civil Time.)	
Jan. 1	6h 41m 19s P.M.	− 3m 56s.8
11	6 1 51 "	56.9
21	5 22 22 "	56.9
31	4 42 53 "	56.9
Feb. 10	4 3 24 "	56.8
20	3 23 56 "	56.7
Mar. 2	2 44 29 "	56.6
12	2 5 4 "	56.4
22	1 25 41 "	56.2
Apr. 1	12 46 20 "	56.0
11	12 7 1 "	55.8
21	11 27 44 A.M.	55.7
May 1	10 48 28 "	55.7
11	10 9 14 "	55.3
21	9 30 2 "	55.1
31	8 50 51 "	55.0
June 10	8 11 41 "	55.0
20	7 32 31 "	54.9
30	6 53 22 "	54.9
July 10	6 14 13 "	54.8
20	5 35 5 "	54.9
30	4 55 56 "	54.9
Aug. 9	4 16 47 "	55.0
19	3 37 37 "	55.1
29	2 58 26 "	55.2
Sept. 8	2 19 14 "	55.3
18	1 40 1 "	55.4
28	1 0 46 "	55.5
Oct. 8	12 21 30 "	55.7
Oct. 13	12 1 51 A.M.	. . . . .
Oct. 13	11 57 55 P.M.	. . . . .
Oct. 17	11 42 12 P.M.	55.8
27	11 2 53 "	56.0
Nov. 6	10 23 32 "	56.1
16	9 44 10 "	56.3
26	9 4 46 "	56.4
Dec. 6	8 25 21 "	56.5
16	7 45 55 "	56.7
26	7 6 27 "	56.8
36	6 26 58 "	− 3 56.9

Local mean time of transit at Washington, May 11, 1902
$= 10^h\ 9^m\ 14^s$ A.M.
Longitude of Cincinnati *west* of Washington
$= +\ 0^h\ 29^m\ 40^s = +\ 0^d.021.$
May $15^d + 0^d.021$ = May $15^d.021$.
Preceding tabular date = May 11.
Therefore, interval $= 4^d.021$.
Daily variation $= -\ 3^m\ 55^s.3 = -\ 235^s.3$.
Total change $= 4.021 \times (-\ 235^s.3) = -\ 15^m\ 46^s$.

$10^h\ \ 9^m\ 14^s$ A.M.
$-\ 15\ \ 46$
$9^h\ 53^m\ 28^s$ A.M.

Therefore, the required time is $9^h\ 53^m\ 28^s$ A.M., May 15, 1902.

SECTION VII

THE Y LEVEL

Description. The essential parts of the Y level (Fig. 18) are, the *telescope*, which is of various lengths, usually about 20 inches, and rests on supports called Y's, from their shape; the *spirit level*, which is under the telescope and attached to it; and the *leveling head* and *tripod*, which are similar to the same parts of the transit.

Leveling Rod. There are several kinds of *leveling rods*, each possessing some merit peculiar to its purpose. The one shown in Fig. 19 is known as the Philadelphia leveling rod, and is the one in most common use. It is made of two pieces of wood, sliding upon each other, and held in position by a clamp. The front surface of each piece is graduated to hundredths of a foot up to 7 feet; the back surface of the rear piece is figured from 7 to 13 feet, reading from the top down,

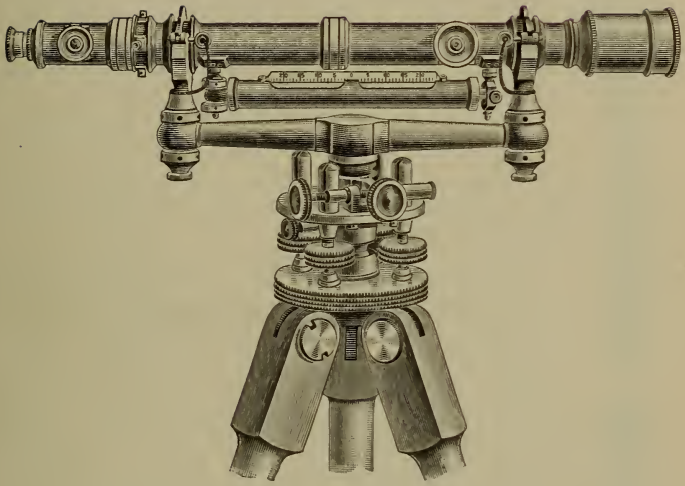


FIG. 18. THE Y LEVEL

and it also has a scale by which the rod is read to half hundredths of a foot as it is extended. The target slides along the front of the rod and is held in place by two springs which press upon the sides of the rod. It has a square opening at the centre, through which the division line of the rod opposite to the horizontal line of the target may be seen. It also carries a scale by which heights may be read to half hundredths of a foot. For heights not greater than 7 feet, the target is moved up or down the front surface, the rod being closed and clamped; but when a greater height is required the target is fixed at 7 feet and the rear half of the rod extended to the required height. The rod thus becomes a self-reading rod 13 feet long.

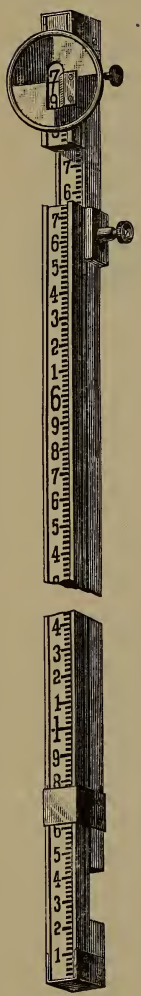


Fig. 19

How to use Level and Rod. When the leveling instrument is used, the tripod should be set firm; the spirit level should then be brought successively over each opposite pair of leveling screws and leveled in each position, the operation being repeated until the bubble remains in the middle of the tube through an entire rotation of the telescope. Each time before taking an observation the instrument should be examined to see if it is still level. Care should be taken to bring the cross wires of the telescope precisely in focus and the object into such perfect view that the wires will appear to be fastened to the surface, however the eye is moved. For very accurate work the instrument should be shielded from the direct rays of the sun.

The leveling rod should be held in a truly vertical position, the rodman standing squarely behind it.

The target is then raised or lowered at the signal of the leveler until its horizontal line is cut by the intersection of the cross wires of the telescope. The reading is done by the leveler or the rodman according to the kind of rod used.

Substitutes for the Y Level. For ordinary work, *the Surveyor's* or *Engineer's Transit* is often used.

The plumb level (Fig. 20) consists of two pieces of wood joined at right angles. A straight line is drawn on the upright perpendicular to the upper edge of the crosshead. The instrument is fastened to a support by a screw through the centre of the crosshead. The upper edge of the crosshead is brought to a level by making the line on the upright coincide with a plumb line.

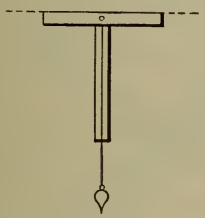


FIG. 20

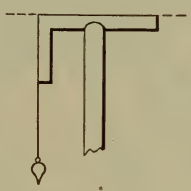


FIG. 21

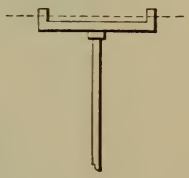


FIG. 22

A carpenter's square can be made into a level by being supported by a post (Fig. 21), the top of which is split or sawed so as to receive the longer arm. The shorter arm is made vertical by a plumb line, which brings the longer arm to a level.

The water level, as shown in Fig. 22, consists of two upright glass tubes cemented into a connecting tube of any material. The whole is nearly filled with water and supported at a convenient height. The surface of the water in the uprights determines the level. The water should be colored.

A level line may be obtained by sighting along the upper surface of the block in which an ordinary *spirit level* is mounted.

For many purposes not requiring great accuracy, any of the foregoing simple instruments in connection with any graduated rod will be sufficient.

EXERCISE IV

1. Set up the level and take the readings on the leveling rod at two stations equally distant from the instrument. What does the difference of these readings indicate?

2. Set up the level successively at the two stations in Example 1, taking the readings on the leveling rod placed where the instrument was first. What does the difference of these readings indicate? Ought this difference to be the same as that in Example 1? Explain.

3. In the field of Example 1, Exercise II, p. 230, set up the level successively at the middle of each of the five sides, taking the readings on the rod each time at both adjacent stations of the field. Find the difference between the sum of the hind-sights and the sum of the foresights. What should this difference equal?

SECTION VIII

THE PLANE TABLE

Description and Uses. The *plane table*, an approved form of which is shown in Fig. 23, consists mainly of a drawing board made of well-seasoned wood, arranged in sections to prevent warping, and supported at a convenient height by a *tripod* and *leveling head*, with attachments for horizontal movement.

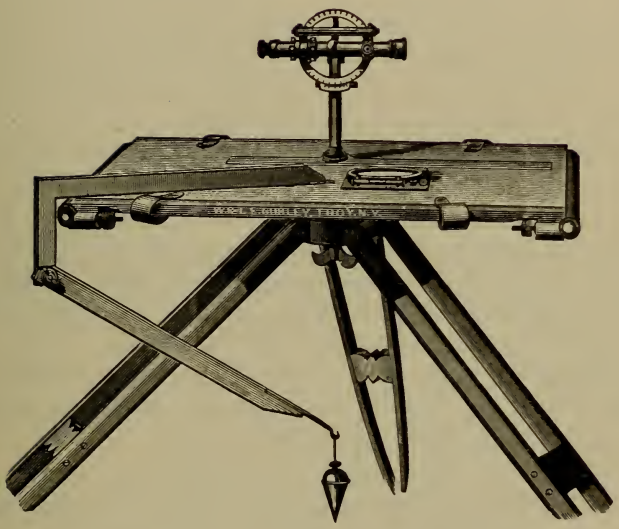


FIG. 23. THE PLANE TABLE

The board is provided with rollers or clamps or both, for keeping the paper secure and even. The *plumbing arm* has its end brought to a point which, however placed on the paper, is directly above the corresponding point on the ground determined by the plummet. The *alidade* is a ruler of brass or steel supporting a telescope with stadia or sight standards, whose line of sight is in or parallel to the same vertical plane with the beveled edge of the ruler. A compass with two spirit levels serves both to level the table and, when applied by the edges parallel to the zero line of the compass circle, to determine the magnetic bearing of the lines drawn on the paper, or the direction of the table itself.

After the principal lines of a survey have been determined and plotted, the details of the plot may be filled in by means of the plane table; or, when a plot only of a tract of land is desired and extreme accuracy is not required, this instrument affords the most expeditious means of obtaining it. There is little use for it outside of the United States Coast and Geodetic Survey and the United States Geological Survey.

To orient the Table. This operation consists in placing the table so that the lines of the plot shall be parallel to the corresponding lines on the ground.

This may be accomplished approximately by turning the table until the needle of the compass indicates the same bearing as at a previous station, the edge of the compass coinciding with the same line on the paper at both stations.

If, however, the line connecting the station at which the instrument is placed with another station is already plotted, the table may be placed in position accurately by placing it over the station so that the plotted line is by estimation over and in the direction of the line on the ground; then making the edge of the ruler coincide with the plotted line, and turning the board until the line of sight bisects the signal at the other end of the line on the ground.

To plot any Point. Let ab on the paper represent the line AB on the ground; it is required to plot c, representing C on the ground.

1. *By intersection.*

Place the table in position at A (Fig. 24), plumbing a over A, and making the fiducial edge of the ruler pass through a; turn the alidade about a until the line of sight bisects the signal at C, and draw a line along the fiducial edge of the ruler. Place the table in position at B, plumbing b over B, and repeat the operation just described. Then c is the intersection of the two lines thus drawn.

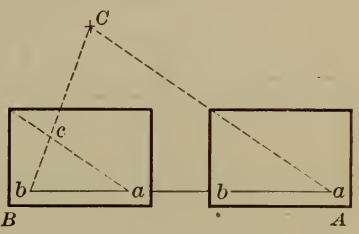


FIG. 24

2. *By resection.*

Place the table in position at A (Fig. 25), and draw a line in the direction of C, as in the former case; then remove the instrument to C, place it in position by the line drawn from a, make the edge of the ruler pass through b, and turn the alidade about b until B is in the line of sight. A line drawn along the edge of the ruler will intersect the line from a in c.

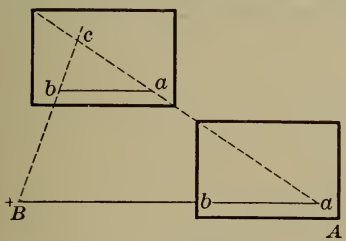


FIG. 25

3. *By radiation.*

Place the table in position at A (Fig. 26), and draw a line from a toward C, as in the former cases. Measure AC, and lay off ac to the same scale as ab.

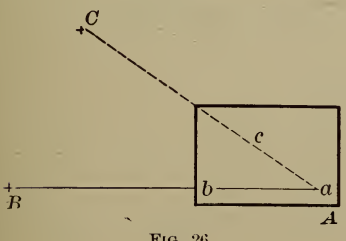


FIG. 26

To plot a Field $ABCD \cdots$

By radiation.

Set up the table at any point P, and mark p on the paper over P. Draw indefinite lines from p

toward A, B, C, $\cdots$ Measure PA, PB, $\cdots$, and lay off pa, pb, $\cdots$ to a suitable scale, and join a and b, b and c, c and d, $\cdots$

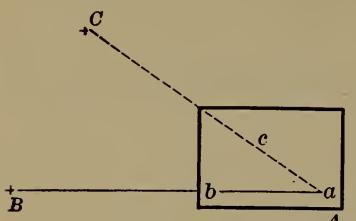


FIG. 27

By progression.

Set up the table at A, and draw a line from a toward B. Measure AB, and plot ab to a suitable scale. Set up the table in position at B, and in like manner determine and plot bc; and so on.

By intersection.

Plot one side as a base line. Plot the other corners by the method of intersection, and join these points in proper order by straight lines.

By resection.

Plot one side as a base line. Plot the other corners by the method of resection, and join these points in proper order by straight lines.

The Three-Point Problem. Let A, B, C represent three field stations plotted as a, b, c, respectively (Fig. 28); it is required

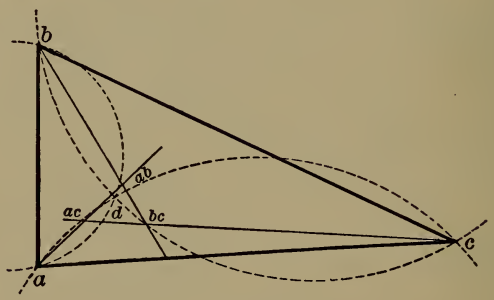


FIG. 28

to plot d representing a fourth field station D, from which A, B, and C are visible.

Place the table over D, level and orient approximately by the compass. Determine d by resection as follows: Make the

edge of the ruler pass through a and lie in the direction aA, and draw a line along the edge of the ruler. In like manner, draw lines through b toward B and through c toward C. If the table is oriented perfectly, these lines meet at the required point d, but ordinarily they will form the *triangle of error*, ab, ac, bc. In this case, through a, b, and ab; a, c, and ac; and b, c, and bc, respectively, draw circles; these circles will intersect in the required point d. For at the required point the sides ab, ac, bc must subtend the same angle as at the points ab, ac, bc, respectively. Hence, the required point d lies at the intersection of the three circles mentioned. The plane table may now be oriented accurately.

The three-point problem may also be solved by fastening on the board a piece of tracing paper and marking the point d representing D, after which lines are drawn from d toward A, B, and C. The tracing paper is then moved until the lines thus drawn pass through a, b, c, respectively, when by pricking through d the point is determined on the plot below. This method, however, is impracticable in case the wind blows.

CHAPTER II

OFFICE INSTRUMENTS

SECTION IX

PLOTTING INSTRUMENTS

Definitions. A *map* is a representation by means of points, lines, and conventional signs on a plane surface, as on paper of a surveyed portion of the earth's surface, including objects upon it. If only the boundary lines are drawn, the representation is called an outline map, or *plot*. The plot is a figure similar to the original, and the ratio of a line of the field to the corresponding line of the plot is called the *scale*. In surveying it is customary to designate the scale as so many chains to the inch.

Principal Minor Instruments. The principal minor instruments used in plotting are a *ruler, pencil, straight-line pen, hair-spring dividers, compasses,* a *right triangle* of wood or hard rubber, a *T-square,* and a *parallel ruler.*

The Diagonal Scale. A portion of this scale is shown in Fig. 29. AB is the unit. AB and $A'B'$ are divided into ten equal parts, and B is joined with h, the first division point to the left of B'; the first division point to the left of B is joined with the second to the left of B', and so on. The part of the horizontal line numbered 1 intercepted between BB' and Bh is evidently $\frac{1}{10}$ of $\frac{1}{10} = \frac{1}{100}$ of the unit; the part of the horizontal line numbered 2 intercepted between BB' and Bh is $\frac{2}{100}$ of the unit, and so on.

The method of using this scale is as follows:
Let it be required to lay off the distance 1.43.

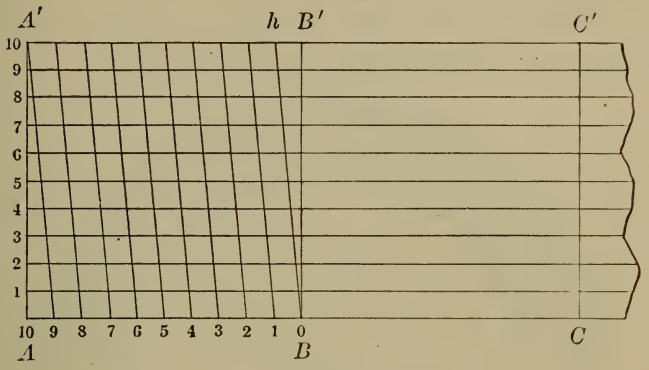


FIG. 29

Place one foot of the dividers at the intersection of the horizontal line numbered 3 and the diagonal numbered 4, and place the other foot at the intersection of the vertical line numbered 1 (CC') and the horizontal line numbered 3; the distance between the feet of the dividers will be the distance required. For, measuring along the horizontal line numbered 3, from CC' to BB' is 1; from BB' to Bh is 0.03; and from Bh to the diagonal numbered 4 is 0.4; and $1 + 0.03 + 0.4 = 1.43$.

The Circular Protractor. This instrument (Fig. 30) usually consists of a semicircular piece of brass or german silver, with its arc divided into degrees and its centre marked.

Some protractors have an arm which carries a vernier, by which angles may be constructed to single minutes. Still others embrace an entire circle and have several arms with verniers.

A *rectangular protractor*, having the degrees marked off on three sides of a plane scale, is sometimes used. Often this form of the protractor is found on the reverse side of the diagonal scale.

Constructions. 1. *To lay off an angle with the circular protractor.* Place the centre over the vertex of the angle, and make the diameter coincide with the given side of the angle. Mark off the number of degrees in the given angle, and draw a line through this point and the vertex.

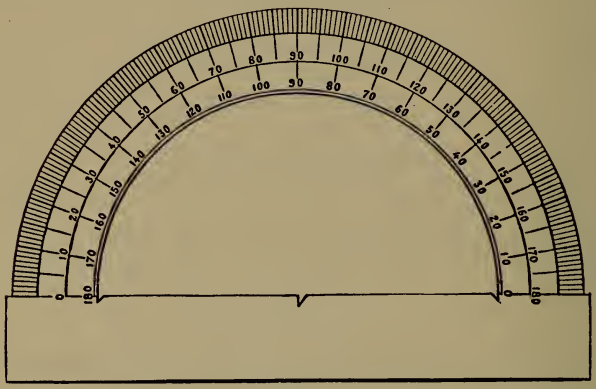


FIG. 30

2. *To draw through a given point a line parallel to a given line with a right triangle and ruler.*

Make one of the sides of the triangle coincide with the given line, and, placing the ruler against one of the other sides, move the triangle along the ruler until the first side passes through the given point; then draw a line along this side.

3. *To draw through a given point a line perpendicular to a given line with a right triangle and ruler.*

Make the hypotenuse of the right triangle coincide with the given line, and, placing a ruler against one of the other sides of the triangle, revolve the triangle about the vertex of the right angle as a centre until its other perpendicular side is against the ruler; then move the triangle along the ruler until the hypotenuse passes through the given point, and draw a line along the hypotenuse.

SECTION X

COMPUTING INSTRUMENTS

The Planimeter. This is an instrument for measuring the area of any irregular field, by applying it to a plot of the field drawn accurately to scale. The form in most common use is that known as the polar planimeter. The essential parts are two arms, one fixed in length, the other adjustable, and a rolling wheel mounted on an axis parallel to the adjustable arm. The outer end of the arm of fixed length is made fast to the plot by means of a needle point, and the free end of the other arm is made to trace the perimeter of the figure to be measured. A disk records the area in the unit for which the instrument is set.

The Slide Rule. This is an instrument for effecting the processes of multiplication, division, involution, and evolution by means of logarithms. It consists of a series of scales so arranged that by sliding one upon the other the addition or subtraction of logarithms is mechanically performed. For a full description of this labor-saving device in its various forms, the student is referred to some treatise on the subject.

CHAPTER III

LAND SURVEYING

SECTION XI

DEFINITIONS

Land Surveying is the art of measuring, laying out, and dividing land, computing parts and areas from measured parts, and preparing a plot. An *original survey* includes laying out the boundary lines and establishing the corners. A *resurvey* is the retracing of old boundary lines and the finding of corner monuments, or the relocating of them when lost.

Rules for Areas. The unit of land measure is the

$$\textit{acre} = 10 \text{ square chains} = 4 \text{ roods}$$
$$= 160 \text{ square rods, perches, or poles.}$$

Areas are referred to the horizontal plane, no allowance being made for inequalities of surface.

Let A, B, and C be the angles of a triangle, and a, b, and c the opposite sides, respectively, and let $s = \frac{1}{2}(a + b + c)$.

Area of triangle $ABC = \frac{1}{2}$ base $\times$ altitude

$$= \tfrac{1}{2} bc \sin A$$
$$= \frac{a^2 \sin B \sin C}{2 \sin (B + C)}$$
$$= \sqrt{s(s-a)(s-b)(s-c)}.$$

Area of rectangle = base $\times$ altitude.

Area of trapezoid $= \frac{1}{2}$ sum of parallel sides $\times$ altitude.

NOTE. Spanish American units are in use in Texas, California, and Mexico. In this system the vara is the unit of length, which in Texas is

reckoned $33\frac{1}{3}$ inches, in California 33 inches, in Mexico 32.9927 inches. The area of a square 1000 varas on a side is called a *labor*, and of a square 5000 varas on a side is called a *league*.

SECTION XII

SPECIAL METHODS OF SURVEYING, AND COMPUTING AREAS

Triangular Fields. Measure, as may be most convenient, the three sides, two sides and the included angle, two angles and the included side, or a side and the altitude upon that side, and compute the area by the appropriate formula.

Fields having More than Three Straight Sides. Divide the field into triangles and take the sum of the areas of the triangles. Or, run a diagonal and perpendiculars to it from the opposite vertices; take the sum of the areas of the right triangles, rectangles, and trapezoids thus formed.

A third method is as follows: Let $ABCD$ (Fig. 31) represent a field, and P and P' two stations within it. (They may be without the field.) Measure PP' with great exactness. Measure the angles between PP' and the lines from P and P' to the corners of the field.

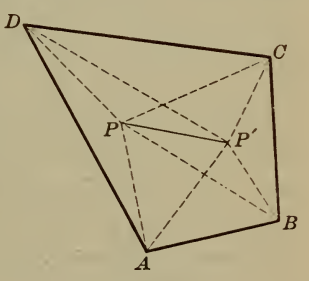


FIG. 31

In the triangle $P'PD$, PP' and the angles $PP'D$ and $P'PD$ are known; hence, PD may be found. In like manner, PC may be found. Then, in the triangle PDC, PD, PC, and the angle DPC are known; hence, the area of PDC may be computed. In like manner, the areas of all the triangles about P or P' may be determined.

$$\text{Area } ABCD = PAD + PDC + PCB + PBA;$$

also, $$\text{area } ABCD = P'AD + P'DC + P'CB + P'BA.$$

Fields having Irregular Boundary Lines. Let $AGBCD$ (Fig. 32) represent a field having a stream $AEFGHKB$ as a boundary line. Run the line AB. From E, F, G, H, K, prominent points on the bank of the stream, let fall perpendiculars EE', FF', GG', etc., upon AB. Regarding AE, EF, etc., as straight

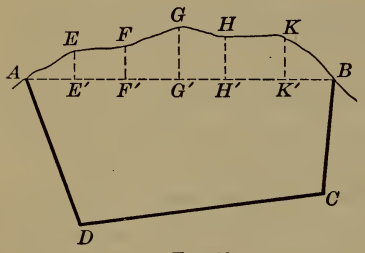


FIG. 32

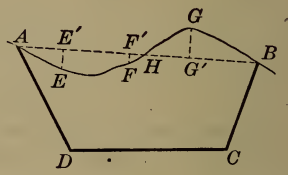


FIG. 33

lines, the portion of the field cut off by AB is divided into right triangles, rectangles, and trapezoids, the necessary elements of which can be measured and the areas computed. The sum of these areas added to the area of $ABCD$ gives the area required. If the offsets are at regular intervals, then the area of the part cut off by AB may be found by adding the offsets and multiplying by the common distance between them.

When the irregular boundary line crosses the straight line that joins its extremities, as in Fig. 33, the areas of $AEFH$ and HGB may be found separately, as in the preceding case. Then, the area required $= ABCD + HGB - AEFH$.

Rectangular System of Co-ordinates. Let XX' and YY' (Fig. 34) be two fixed perpendicular lines intersecting at the point O. Let the four parts into which these lines divide the plane be called Quadrants, as in Trigonometry, and be distinguished by naming them, respectively, *first*, *second*, *third*, and *fourth* quadrants.

Suppose the position of a point is described by saying that its distance from YY', expressed in terms of some chosen unit

of length, is 3, and its distance from XX' is 4. Then there is in each quadrant one point and only one which will satisfy these conditions. The position of the point in each quadrant may be found by drawing parallels to YY' at the distance 3 from YY', and parallels to XX' at the distance 4 from XX'; then the intersections P_1, P_2, P_3, and P_4 satisfy the given conditions.

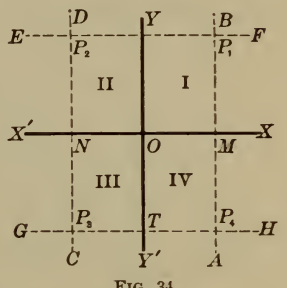


FIG. 34

In order to determine which one of the four points, P_1, P_2, P_3, P_4, is meant, we adopt the rule that distances measured from YY' *to the right* are *positive; to the left, negative.* Distances measured from XX' *upward* are *positive; downward, negative.* Then, the position of P_1 will be denoted by $+3, +4$; of P_2, by $-3, +4$; of P_3, by $-3, -4$; of P_4, by $+3, -4$.

The fixed lines XX' and YY' are called the **Axes of Co-ordinates**; XX' is called the **Axis of Abscissas**, or **Axis of x**; YY', the **Axis of Ordinates**, or **Axis of y**. The intersection O is called the **Origin**.

The two distances (with signs prefixed) which determine the position of a point are called the **Co-ordinates** of the point; the distance of the point from YY' is called its **Abscissa**; and the distance from XX', its **Ordinate**.

Abscissas are usually denoted by x, and ordinates by y, and a point is represented algebraically by simply writing the values of its co-ordinates within parentheses, that of the abscissa being always written first.

Thus, P_1 (Fig. 34) is the point $(3, 4)$, P_2 the point $(-3, 4)$, P_3 the point $(-3, -4)$, and P_4 the point $(3, -4)$. In general the point whose co-ordinates are x and y is the point (x, y).

This system of co-ordinates may be applied to the determination of areas in the following manner:

Suppose the field to be $ABCDE$ (Fig. 35). Lay out the two axes so that the field shall lie wholly within the first quadrant. Then measure the co-ordinates of each of the vertices

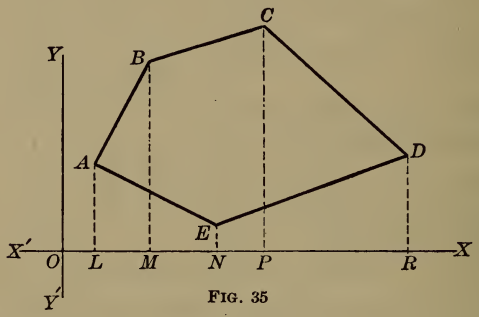


FIG. 35

and designate them as follows: for A, (x_1, y_1); for B, (x_2, y_2); for C, (x_3, y_3); for D, (x_4, y_4); for E, (x_5, y_5). Evidently each of these co-ordinates is positive. Then,

$$\text{area } ABCDE = \text{area } LABM + \text{area } MBCP + \text{area } PCDR - \text{area } NEDR - \text{area } LAEN;$$

or, in terms of the co-ordinates,

$$\begin{aligned}\text{area } ABCDE &= \tfrac{1}{2}(y_1 + y_2)(x_2 - x_1) + \tfrac{1}{2}(y_2 + y_3)(x_3 - x_2) \\ &\quad + \tfrac{1}{2}(y_3 + y_4)(x_4 - x_3) - \tfrac{1}{2}(y_4 + y_5)(x_4 - x_5) \\ &\quad - \tfrac{1}{2}(y_5 + y_1)(x_5 - x_1), \\ &= \tfrac{1}{2}\{x_1(y_5 - y_2) + x_2(y_1 - y_3) + x_3(y_2 - y_4) \\ &\quad + x_4(y_3 - y_5) + x_5(y_4 - y_1)\}.\end{aligned}$$

This method can be put in the form of a general rule:

Take one-half the algebraic sum of the products obtained by multiplying the abscissa of each vertex by the difference between the ordinates of the two adjacent vertices, taken in the clockwise order.

EXERCISE V

1. Required the area of a triangular field whose sides are 13 chains, 14 chains, and 15 chains.

2. Required the area of a triangular field if it has two angles 48° 30′ and 71° 45′, and the included side 20 chains.

3. Required the area of a triangular field whose base is 12.60 chains, and altitude 6.40 chains.

4. Required the area of a triangular field which has two sides 4.50 chains and 3.70 chains, and the included angle 60°.

5. Required the area of a field in the form of a trapezium, one of whose diagonals is 9 chains, and the two perpendiculars upon this diagonal from the opposite vertices 4.50 chains and 3.25 chains.

6. Required the area of the field $ABCDEF$ (Fig. 36), if
$AE = 9.25$ chains, $FF' = 6.40$ chains,
$BE = 13.75$ chains, $DD' = 7$ chains,
$DB = 10$ chains, $CC' = 4$ chains,
and $AA' = 4.75$ chains.

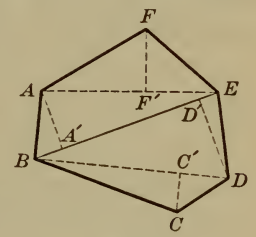


FIG. 36

7. Determine the area of the field $ABCD$ from two interior stations P and P', if $PP' = 1.50$ chains,
$PP'C = 89° 35'$, $PP'D = 349° 45'$, $P'PD = 165° 40'$,
$PP'B = 185° 30'$, $P'PB = 3° 35'$, $P'PC = 303° 15'$.
$PP'A = 309° 15'$, $P'PA = 113° 45'$,

8. Required the area of the field $ABCDEF$ (Fig. 37), if
$AF' = 4$ chains, $FF' = 6$ chains,
$EE' = 6.50$ chains, $AE' = 9$ chains,
$AD = 14$ chains, $AC' = 10$ chains,
$AB' = 6.50$ chains, $BB' = 7$ chains,
$CC' = 6.75$ chains.

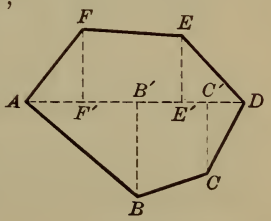


FIG. 37

9. Required the area of the field $AGBCD$ (Fig. 32, p. 260), if the diagonal $AC = 5$, BB' (the perpendicular from B to AC) $= 1$, DD' (the perpendicular from D to AC) $= 1.60$, $EE' = 0.25$, $FF' = 0.25$, $GG' = 0.60$, $HH' = 0.52$, $KK' = 0.54$, $AE' = 0.2$, $E'F' = 0.50$, $F'G' = 0.45$, $G'H' = 0.45$, $H'K' = 0.60$, and $K'B = 0.40$.

10. Required the area of the field $AGBCD$ (Fig. 33, p. 260), if $AD = 3$, $AC = 5$, $AB = 6$, angle $DAC = 45°$, angle $BAC = 30°$, $AE' = 0.75$, $AF' = 2.25$, $AH = 2.53$, $AG' = 3.15$, $EE' = 0.60$, $FF' = 0.40$, and $GG' = 0.75$.

11. Determine the area of the field $ABCD$ from two exterior stations P and P', if $PP' = 1.50$ chains,

$P'PB = 41° 10'$, $P'PD = 104° 45'$, $PP'B = 132° 15'$,
$P'PA = 55° 45'$, $PP'D = 66° 45'$, $PP'A = 103° 0'$.
$P'PC = 77° 20'$, $PP'C = 95° 40'$,

12. Find the area of the field $ABCDE$ (Fig. 35, p. 262), if the co-ordinates, in chains, of the vertices taken in order are (1.40, 6.75), (4.60, 8.32), (9.00, 9.05), (12.15, 5.58), and (5.27, 1.16).

13. Find the area of the field $ABCDE$ (Fig. 35, p. 262), by measuring distances as follows:

$AL = 400$ feet; $BM = 700$ feet; $CP = 680$ feet;
$DR = 380$ feet; $EN = 200$ feet; $LM = 150$ feet;
$MN = 250$ feet; $NP = 200$ feet; $PR = 220$ feet.

14. Lay out a field of four sides, and find its area by the method of triangles and also by the method of rectangular co-ordinates.

15. Lay out a field of six sides, and find its area by the method of triangles and also by the method of rectangular co-ordinates.

SECTION XIII

GENERAL METHOD FOR FARM SURVEYS

Definitions. A *course* is the bearing and length of a line. The *latitude of a course* is the distance between the parallels through its extremities, and is called a *northing* or a *southing*, as the course is northward or southward. The *departure of a course* is the distance between the meridians through its extremities, and is called an *easting* or a *westing*, as the course is eastward or westward. The *meridian distance of a point* is its distance from a meridian. The *double meridian distance of a course* is double the meridian distance of its mid-point, and therefore equal to the sum of the meridian distances of the extremities of the course.

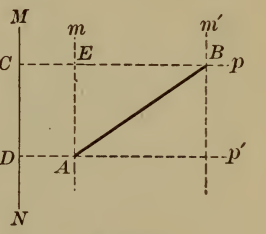


FIG. 38

Let AB (Fig. 38) represent a line, whose bearing and length are known. Let MN be a reference meridian; and let p and p' be parallels through A and B, and m and m' meridians through the same points. Then, angle mAB represents the bearing of line AB. The latitude of the course AB is AE, and its departure EB. The meridian distance of the point B is BC and of A, AD. Evidently, the double meridian distance of the course AB is $\frac{1}{2}(BC + AD)$.

Again, in the triangle AEB,

$$AE = AB \times \cos EAB, \quad \text{and} \quad EB = AB \times \sin EAB.$$

Hence, latitude = distance × cos of bearing, and departure = distance × sin of bearing. From these formulas, the latitude and departure of any course may be found by means of a table of natural sines and cosines. They may be found also

from the Traverse Table, which is merely the tabulated results of the foregoing method for given courses.

Field Notes. The field notes are kept in a book provided for the purpose. The page is commonly ruled in three columns, in the first of which is written the number of the station; in the second, the bearing of the side; and in the third, the length of the side.

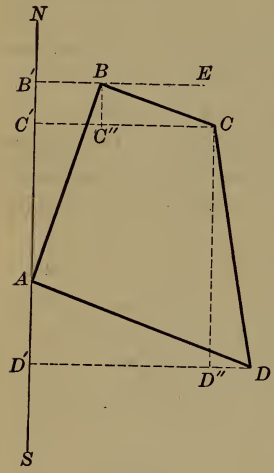


FIG. 39

FIELD NOTES

1	N. 20° E.	8.66
2	S. 70° E.	5.00
3	S. 10° E.	10.00
4	N. 70° W.	10.00

To obtain the field notes, say of field *ABCD* (Fig. 39), place the compass at *A*, the first station, and take the bearing of *AB* (p. 220); suppose it to be N. 20° E. Write the result in the second column of the field notes opposite the number of the station. Measure $AB = 8.66$ chains, and write the result in the third column of the field notes. Place the compass at *B*, and, after testing the bearing of *AB* (p. 221), take the bearing of *BC*, measure *BC*, and write the results in the field notes; and so continue until the bearing and length of each side have been recorded.

Computation of the Area. The survey may begin at any corner of the field; but, for computing the area, the field notes should be arranged so that the most eastern or the most western station shall stand first. For the sake of uniformity, we shall always begin with the most *western* station and keep the field on the *right* in passing around it.

The field notes occupy the first three of the eleven columns in the tablet below. Columns IV, V, VI, and VII contain the latitudes and departures corresponding to the sides, taken from the Traverse Table. The line represented by each number is indicated immediately above that number. Column VIII contains the meridian distances of the points B, C, D, and A, taken in order. Column IX contains the double meridian distances

I	II	III	IV	V	VI	VII	VIII	IX	X	XI
Side	Bearing	Dist.	N.	S.	E.	W.	M.D.	D.M.D.	N.A.	S.A.
			AB'		BB'		BB'	BB'	$2\,ABB'$	
AB	N. 20° E.	8.66	8.14	. . .	2.96	. . .	2.96	2.96	24.0944	. . . .
				$B'C'$	$C''C$		CC'	$BB'+CC'$		$2\,C'CBB'$
BC	S. 70° E.	5.00	. . .	1.71	4.70	. . .	7.66	10.62	. . . .	18.1602
				$C'D'$	$D''D$		DD'	$CC'+DD'$		$2\,D'DCC'$
CD	S. 10° E.	10.00	. . .	9.85	1.74	. . .	9.40	17.06	. . . .	168.0410
			$D'A$			DD'		DD'	$2\,ADD'$	
DA	N. 70° W.	10.00	3.42	. . .	. . .	9.40	0	9.40	32.1480	. . . .
		33.66	11.56	11.56	9.40	9.40			56.2424	186.2012

(186.2012 sq. ch. − 56.2424 sq. ch.) ÷ 2 = 64.98 sq. ch. = 6.50 acres.

of the courses. Their composition is indicated by the letters immediately above the numbers. Column X contains the products of the double meridian distances by the northings in the same line. The first number,

$24.0944 = 2.96 \times 8.14 = BB' \times AB'$ = twice area of triangle ABB';

$32.1480 = 9.40 \times 3.42 = DD' \times AD'$ = twice area of triangle ADD'.

Column XI contains the products of the double meridian distances by the southings in the same line. The first number,

$$18.1602 = 10.62 \times 1.71 = (BB' + CC') \times B'C'$$
$$= \text{twice area of trapezoid } C'CBB';$$
$$168.0410 = 17.06 \times 9.85 = (CC' + DD') \times D'C'$$
$$= \text{twice area of trapezoid } D'DCC'.$$

The sum of the north areas in column X

$$= 56.2424 = 2\,(ABB' + ADD').$$

The sum of the south areas in column XI

$$= 186.2012 = 2\,(C'CBB' + D'DCC').$$

But $(C'CBB' + D'DCC') - (ABB' + ADD') = ABCD.$

Hence, $2\,(C'CBB' + D'DCC') - 2\,(ABB' + ADD') = 2\,ABCD$;

that is, $186.2012 - 56.2424 = 129.9588 = 2\,ABCD.$

Hence, area $ABCD = \frac{1}{2}$ of 129.9588 sq. ch. = 64.98 sq. ch. = 6.50 A.

Balancing the Work. In the survey, we pass entirely around the field; hence, we move just as far north as south. Therefore, the sum of the northings should equal the sum of the southings. In like manner, the sum of the eastings should equal the sum of the westings. In this way the accuracy of the field work may be tested.

In the example on page 267 the sum of the northings is equal to the sum of the southings, being 11.56 in each case; and the sum of the eastings is equal to the sum of the westings, being 9.40 in each case. Hence, the work balances.

In actual practice the work seldom balances. When it does not balance, corrections are generally applied to the latitudes and departures by the following rules:

1. The perimeter of a field is to any one side as the total error in latitude is to the correction required.

2. The perimeter of a field is to any one side as the total error in departure is to the correction required.

Example. The perimeter of a field measured 306.62 chains and one side 72.47 chains, with a total error of 22 links in latitude and of 18 links in departure.

Then $306.62 : 72.47 = 22 \text{ links} : x = 18 \text{ links} : y$.

Whence $x = 5$ links and $y = 4$ links.

Hence the correction in latitude applied to the given side is 0.05 chains, and the correction in departure is 0.04 chains.

If special difficulty was found in taking a particular bearing, or in measuring a particular line, the corrections should be applied to the corresponding latitudes and departures.

The amount of error allowable varies in the practice of different surveyors, and according to the nature of the ground. An error of 1 link in 8 chains would not be considered too great on smooth, level ground; while on rough ground an error of 1 link in 3 chains might be allowed. If the error is considerable, the field measurements should be repeated.

As another example let it be required to find the area of field *ABCDEF* from the following

FIELD NOTES

1	N. 73° 30′ W.	5.00
2	S. 16° 30′ W.	5.00
3	N. 28° 30′ W.	7.07
4	N. 20° 00′ E.	11.18
5	S. 43° 30′ E.	5.00
6	S. 13° 30′ E.	10.00

SIDE	BEARING	DIST.	N.	S.	E.	W.	M.D.	D.M.D.	N.A.	S.A.
							B′B	*B′B*	2 *ABB′*	
AB	N. 20° 00′ E.	11.18	10.51	...	3.82	...	3.82	3.82	40.1482	.....
							C′C	*B′B* + *C′C*		2 *C′C′BB*
BC	S. 43° 30′ E.	5.00	...	3.63	3.44	...	7.26	11.08	.....	40.2204
							D′D	*C′C* + *D′D*		2 *D′DC′C*
CD	S. 13° 30′ E.	10.00	...	9.72	2.33	...	9.59	16.85	.....	163.7826
							E′E	*D′D* + *E′E*	2 *D′DEE′*	
DE	N. 73° 30′ W.	5.00	1.42	...	...	4.79	4.79	14.38	20.4196	.....
						4.80	*F′F*	*E′E* + *F′F*		2 *F′FE′E*
EF	S. 16° 30′ W.	5.00	...	4.79	...	1.42	3.37	8.16	.....	39.0864
								F′F	2 *AFF*	
FA	N. 28° 30′ W.	7.07	6.21	...	...	3.37	0.00	3.37	20.9277	.....
		43.25	18.14	18.14	9.59	9.58 9.59			81.4955	243.0888

$43.25 : 5 = 0.01 : x$. Area = 8.08 acres.

The first station in the field notes is *D*, but we rearrange the numbers in the tablet so that *A* stands first. The northings and southings balance, but the eastings exceed the westings by 1 link. We apply the correction to the westing 4.79 (the distance *DE* being in doubt), making it 4.80, and write the correction. In practice, the corrected numbers are written in red ink, and often all the latitudes and departures are rewritten in four additional columns, headed, respectively, N′, S′, E′, W′.

Supplying Omissions. If for any reason the bearing and the length of any side do not appear in the field notes, the latitude and departure of this side may be found in the following manner :

Find the latitudes and departures of the other sides as usual. The difference between the northings and southings gives the northing or southing of the unknown side, and the difference between the eastings and westings gives the easting or westing of the unknown side.

If the length and the bearing of the unknown side are desired, they may be found by solving the right triangle, whose sides are the latitude and departure found by the method just explained, and whose hypotenuse is the length required.

Obstructions. If the end of a line is not visible from its beginning, or if the line is inaccessible, its length and bearing may be found as follows:

By means of a random line (p. 216).

When it is impossible to run a random line, which is frequently the case on account of the extent of the obstruction, the following method may be used:

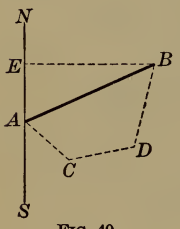


FIG. 40

Let AB (Fig. 40) represent an inaccessible line whose extremities A and B only are known, and B invisible from A.

Set flagstaffs at convenient points, C and D. Find the bearings and lengths of AC, CD, and DB, and then proceed to find the latitude and departure of AB.

EXAMPLE. Suppose that we have the following notes (see Fig. 40):

SIDE	BEARING	DIST.	N.	S.	E.	W.
AC	S. 45° E.	3.00		2.12	2.12	
CD	E.	3.50			3.50	
DB	N. 30° E.	4.83	4.18		2.42	
			4.18	2.12	8.04	0

The northing of AB is $AE = 2.06$, and the easting, $EB = 8.04$. These numbers may be entered in the tablet in the columns N. and E., opposite the side AB.

If the bearing and length of AB are required,

$$\tan BAE = \frac{BE}{AE} = \frac{8.04}{2.06} = 3.903.$$

Hence, the angle $BAE = 75° 38'$.

Also, $$AB = \sqrt{\overline{AE}^2 + \overline{BE}^2} = \sqrt{8.04^2 + 2.06^2} = 8.30.$$

Therefore, the bearing and length of AB are N. 75° 38′ E. and 8.30.

To make a Plot. A plot or map may be drawn to any desired scale. If a line 1 inch in length in the plot represents a line 1 chain in length, the plot is said to be drawn to a scale of 1 chain to an inch. In this case (Fig. 41) the plot is drawn to a scale of 8 chains to an inch.

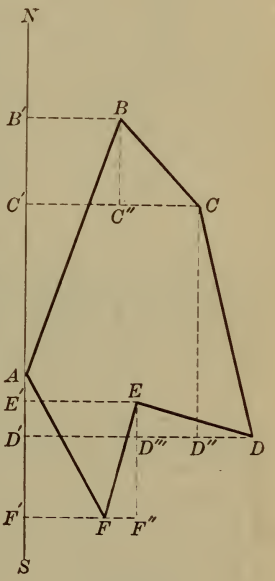


FIG. 41

Draw the line NAS to represent the meridian, and lay off the first northing $AB' = 8.14$. Through B draw an indefinite line perpendicular to NS and lay off $B'B$, the first easting, $= 2.96$. Draw AB; then the line AB represents the first side of the field. Through B draw BC'' perpendicular to BB', and make $BC'' = 1.71$, the first southing. Through C'' draw $C''C$ perpendicular to BC'', and equal to 4.70, the second easting. Draw BC. The line BC represents the second side of the field. Proceed in like manner until the field is completely represented. The extremity of the last line $F'A$, measured from F', should fall at A. This is a test of the accuracy of the plot.

By drawing AC, AE, and EC, the hexagonal figure $ABCDEFA$ is divided into triangles, the bases and altitudes of which may be measured and the area computed approximately.

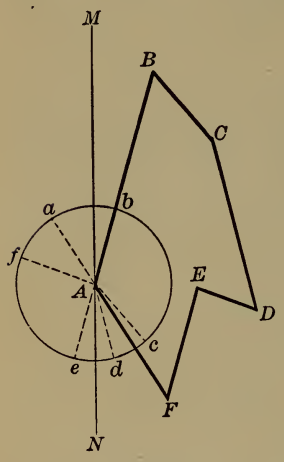


FIG. 42

Another method is as follows: Draw MN (Fig. 42) to represent a meridian. Let the point A in this line be taken as the first station in the rearranged field notes of page 269. With the circular protractor mark off each of the bearings as b, c, d, e, f, and a. Draw AB to scale through b. With triangle and ruler (p. 256) or with parallel ruler draw to scale BC parallel to Ac; and so on.

After some practice, still other methods will be suggested, but the methods given are among the best.

EXERCISE VI

Find the areas of the following and make a plot of each.

In 3 and 7, detours were made on account of obstructions (p. 270). The notes of the detours are written in braces.

1

STA.	BEARINGS	DIST.
1	S. 75° E.	6.00
2	S. 15° E.	4.00
3	S. 75° W.	6.93
4	N. 45° E.	5.00
5	N. 45° W.	5.19½

2

STA.	BEARINGS	DIST.
1	N. 45° E.	10.00
2	S. 75° E.	11.55
3	S. 15° W.	18.21
4	N. 45° W.	19.11

3

STA.	BEARINGS	DIST.
1	S. 2°15′E.	9.68
2 {	N. 51°45′W.	2.39
	S. 85°00′W.	6.47
	S. 55°10′W.	1.62
3	N. 3°45′E.	6.39
4	S. 66°45′E.	1.70
5	N. 15°00′E.	4.98
6	S. 82°45′E.	6.03

4

Sta.	Bearings	Dist.
1	N. 5°30′W.	6.08
2	S. 82°30′W.	6.51
3	S. 3°00′E.	5.33
4	E.	6.72

5

Sta.	Bearings	Dist
1	N. 6°15′W.	6.31
2	S. 81°50′W.	4.06
3	S. 5°00′E.	5.86
4	N. 88°30′E.	4.12

6

Sta.	Bearings	Dist.
1	N. 20°00′E.	4.62½
2	N. 73°00′E.	4.16½
3	S. 45°15′E.	6.18½
4	S. 38°30′W.	8.00
5	Wanting	Wanting

7

Sta.	Bearings	Dist.
1	S. 81°20′W.	4.28
	N. 76°30′W.	2.67
2	N. 5°00′E.	8.68
3	S. 87°30′E.	5.54
4	S. 7°00′E.	1.79
	S. 27°00′E.	1.94
	S. 10°30′E.	5.35
	N. 76°45′W.	1.70

8

Sta.	Bearings	Dist.
1	N. 89°45′E.	4.94
2	S. 7°00′W.	2.30
3	S. 28°00′E.	1.52
4	S. 0°45′E.	2.57
5	N. 84°45′W.	5.11
6	N. 2°30′W.	5.79

9. An Ohio farm is bounded and described as follows: Beginning at the southwest corner of lot No. 13, thence N. $1\frac{1}{4}°$ E. 132 rods and 23 links to a stake in the west boundary line of said lot; thence S. 89° E. 32 rods and $15\frac{4}{10}$ links to a stake; thence N. $1\frac{1}{4}°$ E. 29 rods and 15 links to a stake in the north boundary line of said lot; thence S. 89° E. 61 rods and $18\frac{6}{10}$ links to a stake; thence S. $32\frac{1}{2}°$ W. 54 rods to a stake; thence S. $35\frac{1}{4}°$ E. 22 rods and 4 links to a stake; thence S. 48° E. 33 rods and 2 links to a stake; thence S. $7\frac{1}{2}°$ W. 76 rods and 20 links to a stake in the south boundary line of said lot; thence N. 89° W. 96 rods and 10 links to the place of beginning. Containing 85.87 acres, more or less.

Verify the area given and plot the farm.

Modification of the Latitude and Departure Method. The area of a field may be found by a modification of the latitude and departure method, if its sides and interior angles are known.

Let A, B, C, D represent the interior angles of the field $ABCD$ (Fig. 43). Let the side AB determine the direction of reference. The bearing of AB, with reference to AB, is 0°. The bearing of BC, with reference to AB, is the angle $b = 180° - B$. The bearing of CD, with reference to AB, is the angle $c = C - b$. The bearing of DA, with reference to AB, is the angle $d = A$.

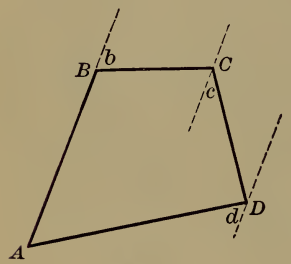


FIG. 43

The area may now be computed by the latitude and departure method, regarding AB as the meridian.

In practice, the exterior angles, when acute, are usually measured. As the interior angles may be measured with considerable accuracy by the transit, the latitudes and departures should be obtained by using a table of natural sines and cosines.

EXERCISE VII

1. Find the area of the field $ABCD$, in which the angle $A = 120°$, $B = 60°$, $C = 150°$, and $D = 30°$; and the side $AB = 4$ chains, $BC = 4$ chains, $CD = 6.928$ chains, and $DA = 8$ chains.

Keep three decimal places, and use the Traverse Table.

2. Find the area of the farm $ABCDE$, in which the angle $A = 106° 19'$, $B = 99° 40'$, $C = 120° 20'$, $D = 86° 8'$, and $E = 127° 33'$; and the side $AB = 79.86$ rods, $BC = 121.13$ rods, $CD = 90$ rods, $DE = 100.65$ rods, and $EA = 100$ rods.

Use the table of natural sines and cosines, keeping two decimal places in the results.

General Remarks on determining Areas. Operations depending upon the reading of the magnetic needle must lack accuracy. Hence, when great accuracy is required (which is seldom the case in land surveying) the method of pp. 266–269 cannot be employed.

The best results are obtained by the methods explained on pp. 259–262 and 274, the horizontal angles being measured with the transit, and great care exercised in measuring the lines.

SECTION XIV

LOCATION SURVEYS

Definition. In surveying proper we measure lines and angles as we find them, while in *location surveys* we mark them out on the ground where they are required to be in order to inclose a given area, or conform to a specified shape, or meet some other given condition. Laying out, parting off, and dividing up land are included in this class of surveys. The surveyor must, for the most part, depend on his general knowledge of Geometry and Trigonometry, and his own ingenuity, for the solutions of problems that arise in location surveys.

Illustrative Problems. Problem 1. To divide a triangular field into two parts having a given ratio, by a line through a given vertex.

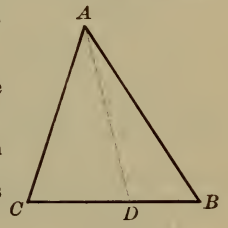


Fig. 44

Let ABC (Fig. 44) be the triangle, and A the given vertex.

Divide BC at D, so that $\frac{BD}{DC}$ equals the given ratio, and draw AD. ABD and ADC are the parts required; for

$$ABD : ADC = BD : DC.$$

Problem 2. To cut off from a triangular field a given area, by a line parallel to the base.

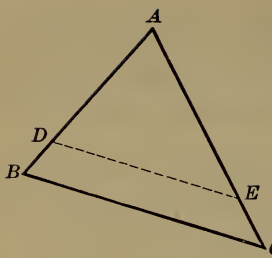


Fig. 45

Let ABC (Fig. 45) be the triangle, and let DE be the division line required.

Then $ABC : ADE = \overline{AB}^2 : \overline{AD}^2$.

$$\therefore \sqrt{ABC} : \sqrt{ADE} = AB : AD.$$

$$\therefore AD = AB\sqrt{\frac{ADE}{ABC}}.$$

Problem 3. To cut off from a triangular field a given fraction of the field, by a line from a given point in a side.

Let ABC (Fig. 46) be the triangle, and P the point from which the line PD is to be located so as to cut off, say, one-third the area of the triangle.

$$AD = AB \times AC \div 3\,AP.$$

For $$ABC : APD = AB \times AC : AP \times AD = 3 : 1.$$

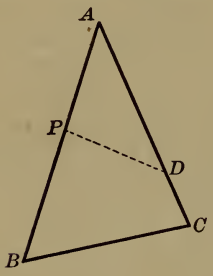


Fig. 46

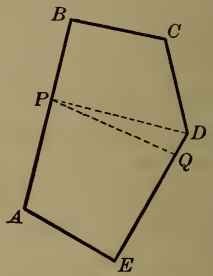


Fig. 47

Problem 4. To divide any field into two parts having a given ratio, by a line through a given point in the perimeter.

Let $ABCDE$ (Fig. 47) represent the field, P the given point, and PQ the required division line.

The areas of the whole field and of the required parts having been determined, run the line PD from P to a corner D, dividing the field, approximately, as required. Determine the area $PBCD$.

The triangle PDQ represents the part which must be added to $PBCD$ to make the required division.

$$\text{Area } PDQ = \tfrac{1}{2} \times PD \times DQ \times \sin PDQ.$$

Hence, $$DQ = \frac{2 \times \text{area } PDQ}{PD \times \sin PDQ}.$$

NOTE. $DQ = \dfrac{2 \times \text{area } PDQ}{\text{perpendicular from } P \text{ on } DE}$. This perpendicular from P on DE may be run and measured directly.

PROBLEM 5. To divide a field into a given number of parts, so that access to a pond of water is given to each.

Let $ABCDE$ (Fig. 48) represent the field, and P the pond. Let it be required to divide the field into four parts. Find the area of the field and of each part.

Let AP be one division line. Run PE, and find the area APE. Take the difference between APE and the area of one of the required parts; this gives the area of the triangle PQE, from which QE may be found, as in Problem 4. Draw PQ; PAQ is one of the required parts. In like manner, PQR and PAS are determined; whence, PSR must be the fourth part required.

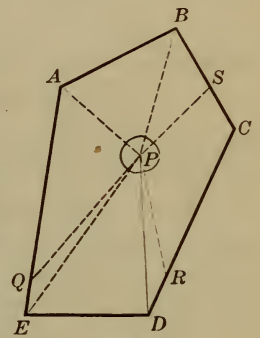


FIG. 48

EXERCISE VIII

1. From the square $ABCD$, containing 6 acres 1 rood 24 perches, part off 3 acres by a line EF parallel to AB.

2. From the rectangle $ABCD$, containing 8 acres 1 rood 24 perches, part off 2 acres 1 rood 32 perches by a line EF parallel to AD which is equal to 7 chains. Then, from the remainder of the rectangle, part off 2 acres 3 roods 25 perches, by a line GH parallel to EB.

3. Part off 6 acres 3 roods 12 perches from a rectangle $ABCD$, containing 15 acres, by a line EF parallel to AB; AD being 10 chains.

4. From a square $ABCD$, whose side is 9 chains, part off a triangle which shall contain 2 acres 1 rood 36 perches, by a line BE drawn from B to the side AD.

5. From $ABCD$, representing the rectangle, whose length is 12.65 chains, and breadth 7.58 chains, part off a trapezoid which shall contain 7 acres 3 roods 24 perches, by a line BE drawn from B to the side DC.

6. In the triangle ABC, $AB = 12$ chains, $AC = 10$ chains, and $BC = 8$ chains; part off a trapezoid of 1 acre 2 roods 16 perches, by the line DE parallel to AB.

7. In the triangle ABC, $AB = 26$ chains, $AC = 20$ chains, and $BC = 16$ chains; part off a trapezoid of 6 acres 1 rood 24 perches, by the line DE parallel to AB.

8. It is required to divide the triangular field ABC among three persons whose claims are as the numbers 2, 3, and 5, so that they may all have the use of a watering place at C; $AB = 10$ chains, $AC = 6.85$ chains, and $CB = 6.10$ chains.

9. Divide the five-sided field $ABCHE$ among three persons, X, Y, and Z, in proportion to their claims, X paying \$500, Y paying \$750, and Z paying \$1000, so that each may have the use of an interior pond at P, the quality of the land being equal throughout. Given $AB = 8.64$ chains, $BC = 8.27$ chains, $CH = 8.06$ chains, $HE = 6.82$ chains, and $EA = 9.90$ chains. The perpendicular PD upon $AB = 5.60$ chains, PD' upon $BC = 6.08$ chains, PD'' upon $CH = 4.80$ chains, PD''' upon $HE = 5.44$ chains, and PD'''' upon $EA = 5.40$ chains. Assume PH as the divisional fence between the shares of X and Z, it is required to determine the position of the fences PM and PN between the shares of X and Y and between the shares of Y and Z.

10. Divide the triangular field ABC, whose sides AB, AC, and BC are 15, 12, and 10 chains, respectively, into three equal

parts, by fences EG and DF parallel to BC, without finding the area of the field.

11. Divide the triangular field ABC, whose sides AB, BC, and AC are 22, 17, and 15 chains, respectively, among three persons, A, B, and C, by fences parallel to the base AB, so that A may have 3 acres above the line AB, B 4 acres above A's share, and C the remainder.

SECTION XV

LAYING OUT THE PUBLIC LANDS

Reference Lines. The public lands north of the Ohio River and west of the Mississippi are generally laid out in accordance with what is known as the rectangular system of surveying. First, an initial point is selected with great care, and then astronomically established. Through this point a *principal meridian*, or true north and south line, is run by means of the solar compass, or the transit with observations on Polaris ; and also an east and west line, called a *base line.* Crossing the principal meridian at intervals of 24 miles, both north and south of the initial point, are run other east and west lines, called *standard parallels*, or correction lines. Northward from the base line and from each of the standard parallels, at intervals of 24 miles, both ways from the principal meridian, are run true north and south lines, called *guide meridians.* Thus, the land is divided into blocks approximately 24 miles square. Six principal meridians have been established, in addition to which and connected with which there are twenty or more independent meridians in the western states and territories.

Division from Reference Lines; Townships. Within each block parallels to the base line, or to a standard parallel, are run at intervals of 6 miles. These are called *township lines.*

At the same intervals are also run north and south lines, called *range lines.* Thus, the tract would be divided into *townships* exactly 6 miles square if it were not for the convergence of the meridians on account of the curvature of the earth. An east and west series of townships is called a *tier,* and a north and south series is called a *range.* A township is designated by giving the number of the tier north or south of the base line and the number of the range east or west of

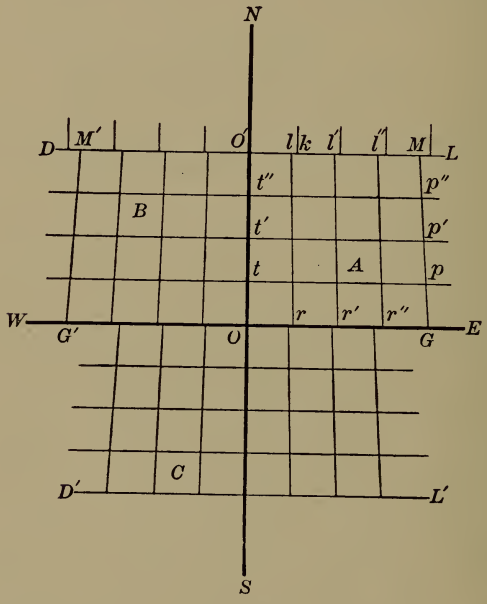


FIG. 49

the principal meridian. Thus, T. 3 N., R. 2 W., read township three north, range two west, means that the township is in the third tier north of the base line, and in the second tier west of the principal meridian.

Let *NS* (Fig. 49) represent a principal meridian; *WE* a base line; *DL* and *D'L'* standard parallels; *GM* and *G'M'* guide

meridians; *rl*, *r'l'*, . . . , range lines; *tp*, *t'p'*, . . . , township lines. If *Or* is taken as 6 miles, then *O'l* will be less than 6 miles. *O'k* being equal to 6 miles and *O'l* being less, it will be observed that there will be offsets on the base line and on standard parallels at intervals of 6 miles.

Township *A* would be designated thus: T. 2 N., R. 3 E. How would townships *B* and *C* be designated?

Subdivision of Townships. The townships are divided into sections approximately 1 mile square, and the sections are divided into quarter sections. The township, section, and quarter-section corners are permanently marked. The sections are numbered, beginning at the northeast corner, as in Fig. 50, which represents a township divided into sections. The quarter sections are designated, according to their position, as N.E., N.W., S.E., and S.W. Section lines are surveyed in such an order as to throw the errors on the northwest quarter sections, which are carefully measured and their areas calculated.

6	5	4	3	2	1
7	8	9	10	11	12
18	17	16	15	14	13
19	20	21	22	23	24
30	29	28	27	26	25
31	32	33	34	35	36

FIG. 50

Meander Lines. If in running a line a navigable stream or a lake more than 1 mile in length is encountered, it is *meandered* by marking the intersection of the line with the bank and running lines from this point along the bank to prominent points which are marked, and the lengths and bearings of the connecting lines recorded.

Manual. For detail of methods, see the "Manual of Surveying Instructions," issued by the Commissioner of the General Land Office, at Washington, D.C., for the use of Surveyors-General.

CHAPTER IV

TRIANGULATION

SECTION XVI

DEFINITIONS

The third method of surveying explained on page 259 is an example of triangulation on a small scale. The simple principle there involved is elaborately worked out in hydrographic or topographic surveys, or in the measurement of terrestrial arcs, as in the "Transcontinental Triangulation and American Arc of the Parallel," recently completed by the United States Coast and Geodetic Survey.

Let F (Fig. 51) represent a point whose position with reference to the base line AB is required. Connect AB with F by the series of triangles ABC, ACD, ADE, and DEF, so that a signal at C is visible from A and B, a signal at D visible from A and C, a signal at E visible from A and D, and a signal at F visible from D and E. In the triangle ABC, the side AB is known, and the angles at A and B may be measured; hence, AC may be computed. In the triangle ACD, AC is known, and the angles at A and C may be measured; hence, AD may be computed. In like manner, DE and EF or DF may be determined. DF, or some suitable line connected with DF, may be measured, and this result

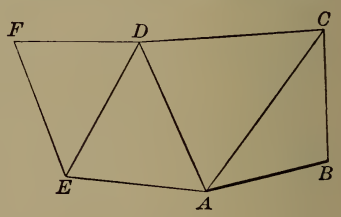


FIG. 51

compared with the computed value to test the accuracy of the field measurement. This net or chain of triangles enables us to determine the relative position of all the points with respect to each other. If the point A is, furthermore, astronomically located, and the azimuth of line AB is known, then we have sufficient data also to determine the absolute geographical position of each of the points.

Classification. Three orders of triangulation are recognized, *viz.: primary*, in which the sides are from 20 to 190 miles in length; *secondary*, in which the sides are from 5 to 40 miles in length, and which connect the primary with the tertiary; *tertiary*, in which the sides are seldom over 5 miles in length, and which bring the survey down to such dimensions as to admit of the minor details being filled in by the compass and plane table.

Measurement of Base Lines. Base lines should be measured with a degree of accuracy corresponding to their importance. Suitable ground must be selected and cleared of all obstructions. Each extremity of the line may be marked by cross lines on the head of a copper tack driven into a stub which is sunk to the surface of the ground. Poles are set up in line about half a mile apart, the alignment being controlled by a transit or theodolite placed over one end of the line. The *preliminary measurement* may be made with an iron wire about one-eighth of an inch in diameter and 60 meters in length, or with a steel chain of the same length.

The *final measurement* is made with the tape line, or with bars 6 meters long, which are supported upon trestles when in use. These bars are placed end to end, and brought to a horizontal position, if this can be quickly accomplished; if not, the angle of inclination is taken by a sector, or a vertical offset is measured with the aid of a transit, so that the exact horizontal distance can be computed. A thermometer is attached to each bar, so that the temperature of the bar may

be noted and a correction for temperature applied. Sometimes the bars are laid in melting ice, in which case accuracy to at least one five-millionth part of the length measured is attainable.

Measurement of Angles. Angles are measured by means of the transit with much greater accuracy than with the compass, since the reading of the plates of the transit is taken to minutes, and by means of microscopes to seconds, while the reading of the needle of the compass is to quarter or half-quarter degrees.

In order to eliminate errors of observation and of adjustment, and errors arising from imperfect graduation of the circles, a large number of readings is made and their mean taken. Two methods are in use, *viz.*, repetition and series.

The method of *repetition* consists essentially in taking as many readings of an angle as is desired, the reading in each case after the first being from the index of the next preceding reading, and then taking the mean.

The method of *series* is the one generally used when several angles about the same point are to be measured. It consists essentially in taking the readings successively on each station, then reversing the telescope and repeating the observations in the reverse order, which completes a series. This process is repeated a number of times, each series beginning with a different index. Then the mean of the different series is found.

On account of the curvature of the earth, the sum of the three angles of a triangle upon its surface exceeds 180°. This *spherical excess*, as it is called, becomes appreciable only when the sides of the triangle are about 5 miles in length. To determine the angles of the rectilinear triangle having the same vertices, one-third of the spherical excess is generally deducted from each spherical angle.

CHAPTER V

LEVELING

SECTION XVII

DEFINITIONS

A *level surface* is a surface parallel with the surface of still water, and is, therefore, slightly curved owing to the spheroidal shape of the earth. A *level line* is a line in a level surface. The *line of apparent level* of a place is a tangent to the level line at that place. Hence, the line of apparent level is perpendicular to the plumb line.

Leveling is the process of finding the difference of level of two places, or the distance of one place above or below a level line through another place.

Corrections for Curvature and Refraction. In ordinary leveling no distinction is made between true and apparent levels. In precise leveling the difference between the two is measured, *i.e.*, correction is made for curvature of the earth. There is sometimes also a correction made for refraction of light.

Let t (Fig. 52) represent the line of apparent level of the place P, a the level line, d the diameter of the earth; then c represents the correction for curvature. To compute the correction for curvature:

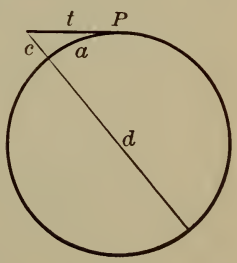


FIG. 52

$$t^2 = c(c + d). \quad (\textit{Geometry}, \S\ 381.)$$

Therefore, $c = \dfrac{t^2}{c + d} = \dfrac{a^2}{d}$, approximately, since c is very small compared with d, and $t = a$, very nearly.

Since d is constant ($= 7920$ miles, nearly), the correction for curvature varies as the square of the distance.

EXAMPLE. What is the correction for curvature for 1 mile?

By substituting in the formula deduced above,

$$c = \frac{a^2}{d} = \frac{1^2}{7920} \text{ miles} = 8 \text{ inches, nearly.}$$

Hence, the correction for curvature for any distance may be found in inches, approximately, by multiplying 8 by the square of the distance expressed in miles.

If correction for refraction is also made, it is customary to diminish the above by about one-sixth of itself; or, $c = \frac{5}{6}$ of $8a^2$.

SECTION XVIII

DIFFERENTIAL LEVELING

Single Setting of Instrument. To find the difference of level between two places when both are visible from some intermediate point, and the difference of level does not exceed 13 feet, only one setting of the level will usually be necessary.

Let A and B (Fig. 53) represent the two places. Set the Y level at a station equally distant, or nearly so, from A and

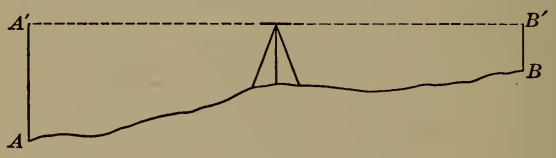


FIG. 53

B, but not necessarily on the line AB. After leveling the instrument, bring the telescope to bear upon the rod (p. 246), and by signal direct the rodman to move the target until its horizontal line is in the line of apparent level of the telescope.

Let the rodman now record the height AA' of the target. In like manner find BB'. The difference between AA' and BB' is the difference of level required. If the instrument is equally distant from A and B, or nearly so, the curvature and the refraction on the two sides of the instrument balance, and no correction for curvature or refraction is necessary.

Several Settings of Instrument. When both places are not visible from the same place, or when the difference of level between them is considerable, two or more settings of the level may be necessary.

Let A and D (Fig. 54) represent the two places. Place the level midway between A and some intermediate station B.

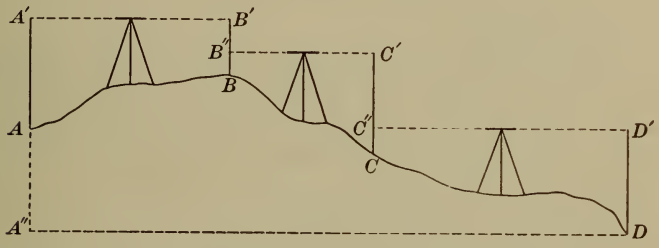


FIG. 54

Find AA' and BB', as in the preceding case, and record the former as a *backsight* and the latter as a *foresight.* Select another intermediate station C, and in like manner find the backsight BB'' and the foresight CC'; and so continue until the place D is reached.

The difference between the sum of the foresights and the sum of the backsights will be the difference of level required.

Since, $BB' + CC' + DD' - (AA' + BB'' + CC'')$

$= BB' - BB'' + CC' - CC'' + DD' - AA'$

$= B'B'' + C'C'' + D'D - AA' = A'A'' - AA' = AA''.$

SECTION XIX

PROFILE LEVELING

Definitions. The intersection of a vertical plane with the surface of the earth is called a *section*, or *profile*. The term "profile," however, usually designates the *plot*, or representation of the section on paper.

Profile leveling is leveling to obtain the data necessary for making a profile or plot of any required section.

A profile is made for the purpose of exhibiting in a single view the inequalities of the surface of the ground for great distances along the line of some proposed improvement, such as a railroad, canal, or ditch, thus facilitating the establishment of the proper grades.

The data necessary for making a profile of any required section are the heights of its different points above some assumed horizontal plane, called the *datum plane*, together with their horizontal distances apart or their distances from the beginning of the section.

The position of the datum plane is fixed with reference to some permanent object near the beginning of the section, called a *bench mark*, and in order to avoid negative heights is assumed at such a distance below this mark that all the points of the section shall be above it.

The heights of the different points of the section above the datum plane are determined by means of the level and leveling rod; and the horizontal length of the section is measured with an engineer's chain or tape, and divided into equal parts, usually 100 feet in length, called *stations*, marked by stakes numbered 0, 1, 2, 3, and so on.

Where the ground is very irregular, it may be necessary, besides taking sights at the regular stakes, to take occasional sights at points between them. If, for instance, at a point

40 feet in advance of stake 3 (Fig. 55) there is a sudden rise or fall in the surface, the height of this point would be determined and recorded as at stake 3.40.

The readings of the rod are ordinarily taken to the nearest tenth of a foot, except on *bench marks* and points called *turning points*, where they are taken to thousandths of a foot.

A *turning point* is a point on which the last sight is taken just before changing the position of the level, and the first sight from the new position of the instrument. A turning point may be coincident with one of the stakes, but must always be a hard point, so that the foot of the rod may stand at the same level for both readings.

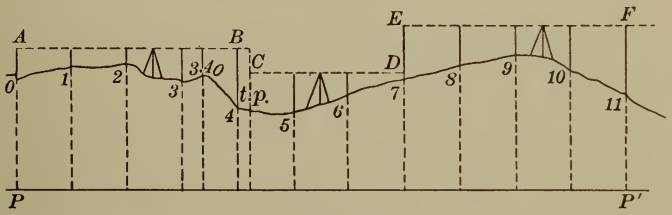


FIG. 55

Field Work. To explain the method of obtaining the field notes necessary for making a profile, let 0, 1, 2, 3, ···, 11 (Fig. 55) represent a portion of a section to be leveled and plotted. Establish a bench mark at or near the beginning of the line, measure the horizontal length of the section, and set stakes 100 feet apart, numbering them 0, 1, 2, 3, and so on. Place the level at some point, as between 2 and 3, and take the reading of the rod on the bench = 4.832. Let PP' represent the datum plane, say 15 feet below the bench mark; then

$$15 + 4.832 = 19.832$$

is the height of the line of sight AB, called the *height of the instrument*, above the datum plane.

Now take the reading at $0 = 5.2 = 0$ A, and subtract the same from 19.832, which leaves $14.6 = 0$ P, the height of the point 0 above the datum plane. Next take sights at 1, 2, 3, 3.40, and 4, equal, respectively, to 3.7, 3.0, 5.1, 4.8, and 8.3, and subtract the same from 19.832; the remainders 16.1, 16.8,

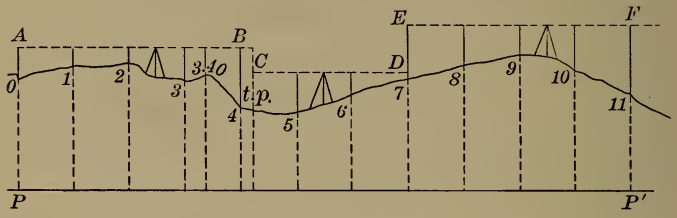


FIG. 56

14.7, 15.0, and 11.5 are respective heights of the points 1, 2, 3, 3.40, and 4.

Then, as it is necessary to change the position of the instrument, select a point in the neighborhood of 4 suitable as a turning point (*t.p.* in the figure), and take a careful reading on it $= 8.480$; subtract this from 19.832, and the remainder, 11.352, is the height of the turning point.

Now carry the instrument forward to a new position, as between 5 and 6, shown in the figure, while the rodman remains at *t.p.*; take a second reading on *t.p.* $= 4.102$, and add it to 11.352, the height of *t.p.* above PP'; the sum 15.454 is the height of the instrument CD in its new position.

Take sight upon 5, 6, and 7, equal, respectively, to 4.9, 2.8, and 0.904; subtract these sights from 15.454, and the results 10.6, 12.7, and 14.550 are the heights of the points 5, 6, and 7, respectively.

The point 7, being suitable, is made a turning point, and the instrument is moved forward to a point between 9 and 10. The sight at $7 = 6.870$, added to the height of 7 gives 21.420 as the height of the instrument EF in its new position. The

readings at 8, 9, 10, and 11, which are, respectively, 5.4, 3.6, 5.8, and 9.0, subtracted from 21.420 give the heights of these points, namely, 16.0, 17.8, 15.6, and 12.4.

Proceed in like manner until the entire section is leveled, establishing bench marks at intervals along the line to serve as reference points for future operations. The bench marks should be described with sufficient minuteness to enable any one not connected with the survey to locate them easily and unmistakably. A record of the work is given in the following table:

STATION	+S.	H.I.	−S.	H.S.	REMARKS
B	4.832			15.0	Bench on rock 20 ft.
0		19.832	5.2	14.6	south of 0
1			3.7	16.1	
2			3.0	16.8	
3			5.1	14.7	3 to 3.40 turnpike road
3.40			4.8	15.0	
4			8.3	11.5	
t.p.	4.102		8.480	11.352	
5		15.454	4.9	10.6	
6			2.8	12.7	
7	6.870		0.904	14.550	
8		21.420	5.4	16.0	
9			3.6	17.8	
10			5 8	15.6	
11			9.0	12.4	
B					Bench on oak stump
12					27 ft. N.E. of 12,
etc.					etc.

The first column contains the numbers or names of all the points on which sights are taken. The second column contains the sight taken on the first bench mark, and the sight on each turning point taken immediately after the instrument

has been moved to a new position. These are called *plus sights* (+ *S.*) because they are *added* to the heights of the points on which they are taken to obtain the height of the instrument given in the third column (*H.I.*). The fourth column contains all the readings except those recorded in the second column. These are called *minus sights* (— *S.*) because they are *subtracted* from the numbers in the third column to obtain all the numbers in the fifth column except the first, which is the assumed depth of the datum plane below the bench. The fifth column (*H.S.*, height of surface) contains the required heights of all the points of the section named in the first column together with the heights of all benches and turning points.

Making the Profile. Draw a line PP' (Fig. 56), to represent the datum plane, and beginning at some point as P, lay off the distances 100, 200, 300, 340, 400 feet, and so on, to the right, using some convenient scale, say 200 feet to the inch. At these points of division erect perpendiculars equal in length to the height of the points 0, 1, 2, 3.40, 4, ···, given in the fifth column of the above field notes, using in this case a larger scale, say 20 feet to the inch. Through the extremities of these perpendiculars draw the irregular line 0, 1, 2, 3, ···, 11, and the result, with some explanatory figures, is the required plot or profile.

The making of a profile is much simplified by the use of *profile paper*, which may be had by the yard or roll.

If a *horizontal plot* is required, the bearings of the different portions of the section must be taken. Such a plot should be made, if it will assist in properly understanding the field work, or if it is desirable for future reference in connection with the field notes. Sometimes both the profile and the plot are drawn side by side on the same sheet; in this case, if the line leveled over is not straight, the profile will be longer than the plot.

SECTION XX

TOPOGRAPHIC LEVELING

The principal object of topographic surveying is to show the contour of the ground. This operation, called topographic leveling, is performed by representing on paper the curved lines in which parallel horizontal planes at uniform distances from each other would meet the surface. It is evident that all points in the intersection of a horizontal plane with the surface of the ground are at the same level. Hence, it is necessary only to find points at the same level and join these to determine a line of intersection.

The method commonly employed will be understood by reference to Fig. 57. The ground $ABCD$ is divided into equal squares, and a numbered stake driven at each intersection. By means of a level and leveling rod the heights of the other stations above m and D, the lowest stations, are determined. A plot of the ground with the intersecting lines is then drawn, and the height of each station written as in the figure.

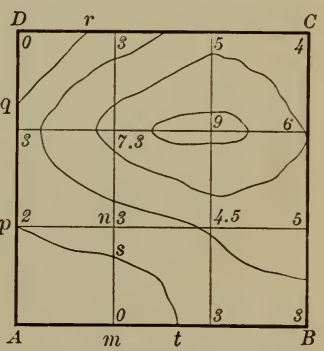


FIG. 57

Suppose that the horizontal planes are 2 feet apart; if the first passes through m and D, the second will pass through p, which is 2 feet above m; and since n is 3 feet above m, the second plane will cut the line mn in a point s determined by the proportion $mn : ms = 3 : 2$. In like manner, the points t, q, and r are determined.

The irregular line $tsp \cdots qr$ represents the intersection of the second horizontal plane with the surface of the ground.

In like manner, the intersections of the planes, respectively, 4, 6, and 8 feet above *m* are traced. The more rapid the change in level the nearer these lines approach each other.

SECTION XXI

DRAINAGE SURVEYING

Preliminaries. The locality to be drained should first be carefully reconnoitered, with the view of ascertaining the general feature of the land so as to enable the surveyor properly to locate the drains; the beginning, route, and terminus of which should all be definitely planned. By the beginning of a drain is meant its highest point.

Field Work. The field work is essentially the same for under drains and for open drains. The first thing is to establish the line of a drain. This includes the setting of stakes at intervals of from 50 feet to 100 feet, and also wherever there is an angle in the line; the bearings and lengths of the successive straight-line sections, beginning with the instrument set over the beginning of the drain; and the designation by distances of the points of meeting of roads and land lines. Levels of the lines are then taken in accordance with the method described on pp. 289–291. If circumstances will permit, it is sometimes of advantage to have the leveling process go hand in hand with the establishing of the line.

Plot and Profile. If a considerable region is to be drained, a plot should be made of the entire tract, and on this plot should be drawn, in proper position, the lines of the drain and its branches. In a suitable place on the sheet should be noted the courses of the various sections of the drain and the number of linear feet belonging to each owner of land within the tract drained. A profile should also be made,

as shown on page 292. From this profile inspection will determine whether a single grade will suffice, or whether a succession of different grades will be better.

EXERCISE IX

1. Find the difference of level of two places from the following field notes: backsights, 5.2, 6.8, and 4.0; foresights, 8.1, 9.5, and 7.9.

2. Stake 0 of the following notes stands at the lowest point of a pond to be drained into a creek; stake 10 stands at the edge of the bank, and 10.25 at the bottom of the creek. Make a profile, draw the grade line through 0 and 10.25, and fill out the columns *H.G.* and *C.*, the former to show the height of grade line above the datum, and the latter, the depth of cut at the several stakes necessary to construct the drain.

STATION	+S.	H.I.	−S.	H.S.	H.G.	C.	REMARKS
B	6.000			25			Bench on rock
0			10.2		20.8	0.0	30 ft. west of
1			5.3			5.3	stake 1
2			4.6				
3			4.0				
4			6.8				
5	4.572		7.090				
6			3.9				
7			2.0				
8			4.9				
9			4.3				
10			4.5				
10.25			11.8				

Horizontal scale, 2 ch. = 1 in.
Vertical scale, 20 ft. = 1 in.

3. Find the difference in altitude between the highest point and the lowest point of the campus or of a field.

4. Obtain the data necessary for a profile of a half mile of highway, and make the profile.

5. Write the proper numbers in the third and fifth columns of the following table of field notes, and make a profile of the section.

STATION	+S.	H.I.	−S.	H.S.	REMARKS
B	6.944			20	Bench on post 22 ft.
0			7.4		north of 0
1			5.6		
2			3.9		
3			4.6		
t.p.	3.855		5.513		
4			4.9		
5			3.5		
6			1.2		

CHAPTER VI

RAILROAD SURVEYING

SECTION XXII

LAYING OUT THE ROUTE

Preliminary Survey. After it has been decided which of several feasible lines is the best, a preliminary survey for final location should be made. This should include, among other things, data referring to elevations, depressions, streams to be crossed, highways, buildings obstructing, character of soil, and natural resources affording materials for construction; also data referring to proximity to towns, titles of land, rights of way, and so on.

Establishing the Roadbed. When the general route of a railroad has been determined, a middle surface line is run with the transit. A profile of this line is determined, as on page 292. The leveling stations are commonly 1 chain (100 feet) apart. Places of different level are connected by a gradient line, which intersects the perpendiculars to the datum line at the leveling stations in points determined by simple proportion. Hence, the distance of each leveling station above or below the level or gradient line which represents the position of the roadbed is known.

SECTION XXIII

CROSS-SECTION WORK

Excavations. If the roadbed lies below the surface, an excavation is made. Let *ACBD* (Fig. 58) represent a cross

section of an excavation, f a point in the middle surface line, f' the corresponding point in the roadbed, and CD the width of the excavation at the bottom. The slopes at the sides are commonly made so that $AA' = \frac{2}{3} A'C$, and $BB' = \frac{2}{3} DB'$. When ff' and CD are known, the points A, B, C', and D' are readily determined by a level and tape measure.

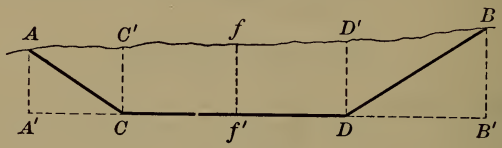


FIG. 58

If from the area of the trapezoid $ABB'A'$ the areas of the triangles $AA'C$ and $BB'D$ are deducted, the remainder is the area of the cross section. In like manner the cross section at the next station may be determined. These two cross sections, if similar, are the bases of a frustum of a quadrangular pyramid whose volume will be the amount of the excavation. In case the cross sections are not similar, the computations should be made by means of the Prismoidal Formula (*Geometry*, § 733).

Embankments. If the roadbed lies above the surface, an embankment is made, the cross section of which is like that of the excavation, but inverted.

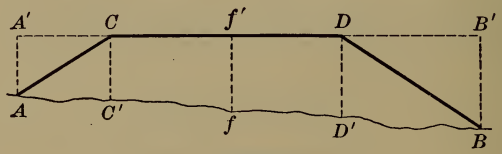


FIG. 59

Fig. 59 represents the cross section of an embankment which is lettered so as to show its relation to the excavation of Fig. 58.

SECTION XXIV

CURVES

Principles. When it is necessary to change the direction of a railroad it is done gradually by a curve, usually the arc of a circle. Let AF and AO (Fig. 60) represent two lines to be thus connected. Take any convenient length $AB = AE = t$. The intersection of the perpendiculars BC and EC determines the centre C, and the radius of curvature $BC = r$. The length of the radius depends on the angle A and the tangent AB. For, in the right triangle ABC,

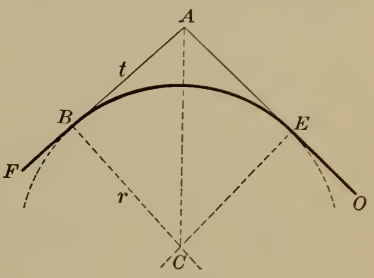


FIG. 60

$$\tan BAC = \frac{BC}{AB}, \quad \text{or } \tan \tfrac{1}{2}A = \frac{r}{t}.$$

Hence, $$r = t \tan \tfrac{1}{2}A.$$

The *degree* of a railroad curve is the angle subtended at the centre of the curve by a chord of 100 feet. If D is the degree of a curve and r its radius,

$$\sin \tfrac{1}{2}D = \frac{50}{r}, \quad \text{and } r = 50 \csc \tfrac{1}{2}D.$$

For example, a 6° curve has a radius of 955.37 feet.

Sometimes the topography of the route is such as to necessitate a successive series of curves of different radii, in which case the whole series of curves is called a *compound curve*, the principles involved being the same for each component as for a simple curve.

Methods of laying out the Curve. 1. Let Bm (Fig. 61) represent a portion of the tangent. It is required to find mP, the perpendicular to the tangent meeting the curve at P.

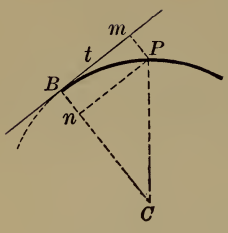


FIG. 61

$$mP = Bn = CB - Cn.$$

But $CB = r$,

and $Cn = \sqrt{\overline{CP}^2 - \overline{Pn}^2}$

$$= \sqrt{r^2 - t^2}.$$

Hence, $mP = r - \sqrt{r^2 - t^2}$.

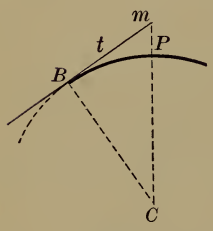


FIG. 62

2. It is required to find mP (Fig. 62) in the direction of the centre.

$$mP = mC - PC.$$

But $mC = \sqrt{\overline{BC}^2 + \overline{Bm}^2} = \sqrt{r^2 + t^2}$.

Hence,

$$mP = \sqrt{r^2 + t^2} - r.$$

3. Place transits at B and E (Fig. 63). Direct the telescope of the former to E, and of the latter to A. Turn each toward the curve the same number of degrees, and mark P, the point of intersection of the lines of sight. P is a point in the circle to which AB and AE are tangents at B and E, respectively.

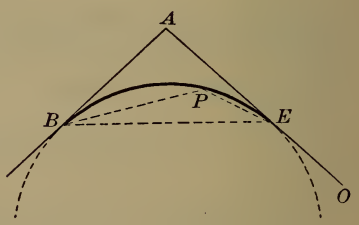


FIG. 63

4. If the degree D of the curve is given and the tangent BA at B (Fig. 64), place the transit at B and direct the telescope toward A. Turn off successively the angles ABP, PBP',

$P'BP''$, $\cdots$, each equal to $\frac{1}{2}D$, and take BP, PP', $P'P''$, $\cdots$, each 100 feet, the length of the tape. Then, P, P', P'', $\cdots$ lie on the required curve.

If the angle A and the tangent distance $BA = t$ are given, D can be found from the formulas

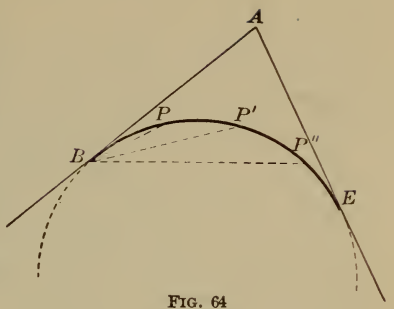


FIG. 64

$$\sin \tfrac{1}{2} D = \frac{50}{r}, \text{ and } r = t \tan \tfrac{1}{2} A.$$

Whence, $$\sin \tfrac{1}{2} D = \frac{50}{t} \cot \tfrac{1}{2} A.$$

EXERCISE X

1. The cross-section areas at five stations, 100 feet apart, of a railroad cut are, respectively, 576.8 square feet, 695.1 square feet, 809.5 square feet, 652.0 square feet, and 511.7 square feet. Compute the volume of material in this portion of the cut: (i) on the hypothesis that the cross sections are similar; (ii) on the hypothesis that they are dissimilar, the alternate cross sections being regarded as mid-sections.

2. Find the radius of a curve of 1°, of 2°, of 3°, of 4°, of 5°.

3. Two adjacent straight sections of a railroad form an angle of 148° 16′. They are joined by a curve touching each of them at the distance of 388 feet from the vertical point. Find the radius and the degree of the curve.

4. Lay out a curve by the first or second method, and check the work by means of one of the transit methods.

CHAPTER VII

CITY SURVEYING

SECTION XXV

FIELD WORK

Instruments. Since the principles in city surveying are essentially the same as those in land surveying, instruments of the same general character as the instruments already described may be used, except that in this class of work the ordinary compass and the chain are set aside. For the smaller cities, an instrument such as the surveyor's transit is sufficient in accuracy for the purposes of angle measurement and for leveling. However, when extreme accuracy is demanded, as in the case of large cities, specially made instruments should be used: a transit reading to 30 seconds, or even to 10 seconds; a high-grade Y level of at least 20-inch length; and a standard tape, tested for sag and temperature.

Streets. In most cases the city engineer must take the streets as he finds them. When a city has outgrown its original plan, if indeed it had any, sheer necessity may demand the location of additional streets or changes in existing streets. If a proposed town or city is to be laid out, the general contour of the ground and location of the site determine to a great extent the system of streets to be adopted. Experience has shown that wherever possible a rectangular system of street lines, with a few well-located diagonal streets, is the most satisfactory. Streets ordinarily vary in width from 50 to 100 feet, and each sidewalk from 7

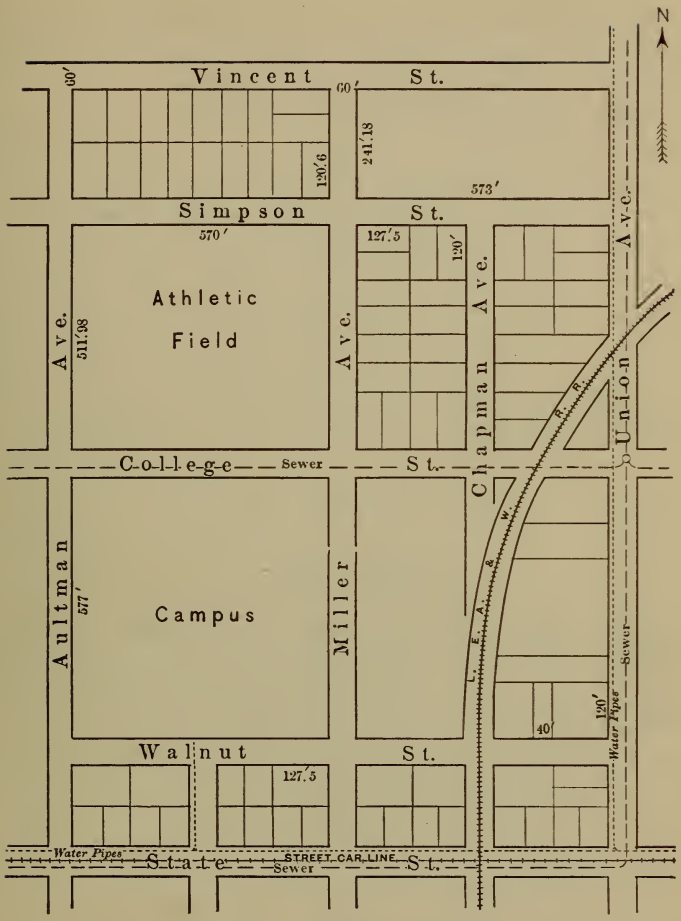


FIG. 65

to 15 feet. The principal improvements of streets are grading, paving, setting curbs, laying sidewalks, constructing sewers, and laying water pipes.

The field work necessary for all these may be included under the heads of leveling, locating lines, and locating points, which have already been described.

Blocks and Lots. There is no established rule for the size of either blocks or lots. Fig. 65 gives some idea of their dimensions. The location of a block is described by reference to the streets which bound it. A lot is described by number and block, or by number alone, or by giving the location and length of its bounding lines. The co-ordinate system of location of points, described on page 261, has much in its favor for use in city surveying. Monuments at points of reference and at intersections of streets and corners of lots should be of permanent character, and set with extreme care.

SECTION XXVI

OFFICE WORK

Plots. Among the more important plots that should be prepared by the city engineer are a complete city map, drawn to scale, showing the streets and alleys, blocks and lots, with dimensions, and the location of railroads, street-car lines, sewage system, water-pipe lines, and so on; a topographical map of the entire city, including as may be found desirable portions of the surrounding region; a profile map of the streets. These are made from the field notes, which should be amply and carefully prepared.

Records. No work of importance, whether done in the field or in the office, should fail to be recorded in some permanent form. Field notes, computations, plots, and copies of work specially prepared should be properly indexed and filed away.

NAVIGATION

CHAPTER I

INTRODUCTION

SECTION I

DEFINITIONS

Navigation is the art of conducting a ship from one port to another port.

The **True Course** of a ship is the angle which her path makes with the true meridian.

The **Correct Magnetic Course** is the angle which the ship's path makes with the magnetic meridian.

When a ship is steered on one course, her path crosses the meridians at the same angle. Such a line is called the **ship's track, rhumb line,** or **loxodromic curve.**

The **Distance** is the length of the path of the ship expressed in geographical or nautical miles.

The two most important problems of navigation are:

(1) To find the present position of the ship.

(2) To determine the future course.

The position (latitude and longitude) of a ship at sea may be found by two distinct methods:

(1) By **Dead Reckoning,** which consists in keeping a record of the courses and distances sailed from a known point, and computing from these data the position reached.

(2) By **Nautical Astronomy,** which consists in finding position by observations of the heavenly bodies, generally the sun.

Problems of the first method except Great Circle Sailing are solved by Plane Trigonometry; problems of the second method are solved by Spherical Trigonometry.

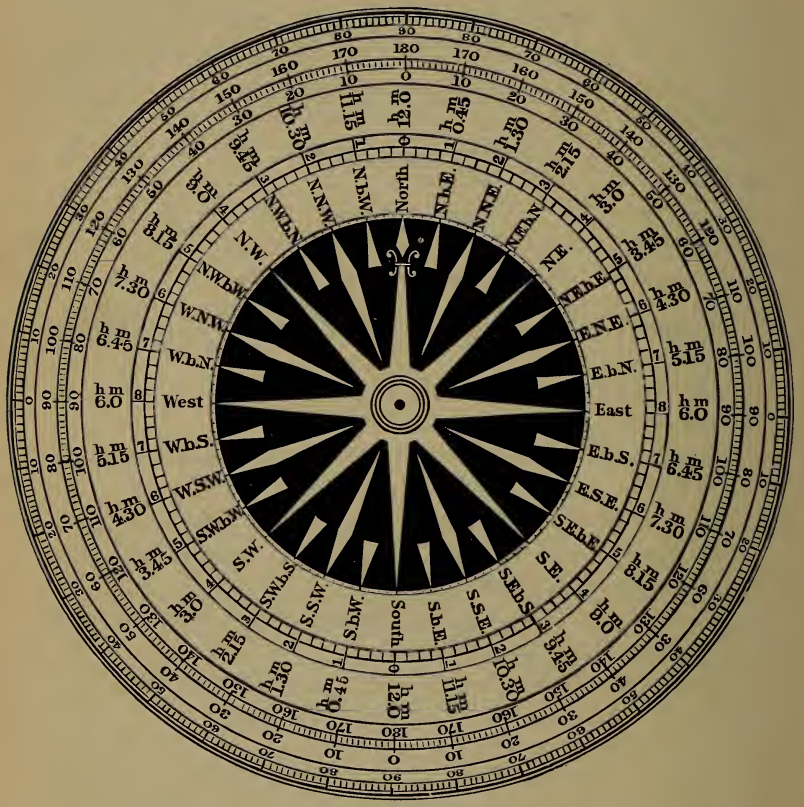


FIG. 1. The Mariner's Compass

To find the position of a ship by observations of celestial bodies requires a knowledge of Nautical Astronomy and Spherical Trigonometry. When a ship is out of sight of land she is steered by the Mariner's Compass.

A Table of the Angles

Which every Point and Quarter Point of the Compass makes with the Meridian

North		Points	° ′ ″	Points	South	
N. by E.	N. by W.	0–¼ 0–½ 0–¾ 1	2 48 45 5 37 30 8 26 15 11 15 0	0–¼ 0–½ 0–¾ 1	S. by E.	S. by W.
N.N.E.	N.N.W.	1–¼ 1–½ 1–¾ 2	14 3 45 16 52 30 19 41 15 22 30 0	1–¼ 1–½ 1–¾ 2	S.S.E.	S.S.W.
N.E. by N.	N.W. by N.	2–¼ 2–½ 2–¾ 3	25 18 45 28 7 30 30 56 15 33 45 0	2–¼ 2–½ 2–¾ 3	S.E. by S.	S.W. by S.
N.E.	N.W.	3–¼ 3–½ 3–¾ 4	36 33 45 39 22 30 42 11 15 45 0 0	3–¼ 3–½ 3–¾ 4	S.E.	S.W.
N.E. by E.	N.W. by W.	4–¼ 4–½ 4–¾ 5	47 48 45 50 37 30 53 26 15 56 15 0	4–¼ 4–½ 4–¾ 5	S.E. by E.	S.W. by W.
E.N.E.	W.N.W.	5–¼ 5–½ 5–¾ 6	59 3 45 61 52 30 64 41 15 67 30 0	5–¼ 5–½ 5–¾ 6	E.S.E.	W.S.W.
E. by N.	W. by N.	6–¼ 6–½ 6–¾ 7	70 18 45 73 7 30 75 56 15 78 45 0	6–¼ 6–½ 6–¾ 7	E. by S.	W. by S.
East	West	7–¼ 7–½ 7–¾ 8	81 33 45 84 22 30 87 11 15 90 0 0	7–¼ 7–½ 7–¾ 8	East	West

The **Card** of the compass consists of a circular plate of mica covered with paper, the circumference of which is usually divided into degrees. An inner circle is divided into thirty-two equal parts by radial lines called *rhumb lines*. The extremities of the rhumb lines are called *points*, or *rhumbs*.

The principle upon which the points are named will be understood by inspecting a single quadrant. The point (N.E.) midway between N. and E. is named by writing these letters together. The point (N.N.E.) midway between N. and N.E. is named by joining these letters. The first point to the east of north is named N. by E., and the first point to

the north of N.E. is named N.E. by N. In like manner, the points between E. and N.E. are named.

Naming the points in order is called **boxing the compass.**

One or more **Magnetic Needles** are attached to the lower surface of the card, parallel to the N. and S. line, having their N. and S. poles in the same direction as the corresponding points of the card.

The card turns upon a pivot in the centre of a cylindrical metallic box called the **Bowl**, which is supported by **gimbals** in a square box. The bowl is hung within a brass hoop, and the hoop itself is hung at points 90° distant from the compass-bowl bearings, so that the card may retain a horizontal position in spite of the pitching and rolling of the ship.

In some compasses the card almost floats in alcohol. This relieves the pivot of the greater part of the weight of the card and needles.

The **Lubber-Line** is a distinct vertical line on the inner side of the bowl.

It is often convenient to refer to the points of the compass as *right* or *left* of *north* or *south.* Suppose an observer to stand at the centre of the card, facing north. Then, all points between N. and E. are right of north (R. of N.), and all points between N. and W. are left of north (L. of N.). In like manner, if the observer faces south, all points between S. and W. are right of south (R. of S.), and all points between S. and E. are left of south (L. of S.).

Thus,
W.S.W. is 6 points, or 67° 30′, R. of S.
N. by W. is 1 point, or 11° 15′, L. of N.
S.E. is 4 points, or 45°, L. of S.
E. by N. is 7 points, or 78° 45′, R. of N.

The compass is placed in a box called the **Binnacle.** When properly placed, the lubber-line is next to the bow of the vessel, and a line passing through the lubber-line and the centre of the card is parallel to the fore-and-aft line of the vessel.

The **Steering Compass** is placed immediately before the helmsman. The **Standard Compass** is commonly placed near the middle of the vessel, in such a position as to be least affected by the iron used in the construction of the vessel. (See Sect. IV, p. 310.)

SECTION II

VARIATION OF THE NEEDLE FROM THE TRUE MERIDIAN

This subject has been treated in Surveying. Variation is found for various places by astronomical observations. The navigator is provided with charts which give the variation at different places.

SECTION III

LOCAL ATTRACTION

The disturbing effect upon the needle by iron outside of the ship (such as the iron in docks) is called *local attraction.*

SECTION IV

DEVIATION

In a ship the needle will not always point to the magnetic north.

Deviation includes the changes in the direction of the needle caused by the iron in the ship.

Deviation is named *easterly* or *westerly* according as the north point of the needle is drawn to the eastward or to the westward of the magnetic north.

The iron used in the construction of a ship consists of two kinds, *soft* iron and *hard* iron. *Soft* iron becomes instantly magnetized to its full capacity when exposed to the influence

of any magnetized body, but loses its magnetism upon the removal of the magnet. This is called *magnetism by induction.*

Hard iron does not become magnetized by ordinary induction, but when magnetized retains its magnetism. Hence, this is called *subpermanent magnetism.* A disturbance of the molecules of the iron by hammering or otherwise is required for it to receive or part with its magnetism.

An iron ship may be correctly regarded as a large permanent magnet, becoming so in the process of construction. The materials used in the construction do not by themselves retain magnetism permanently, but they acquire this property in a greater or less degree when united in the form of a ship and subjected to hammering, etc.

Subpermanent magnetism depends upon the direction of the ship's head while building, and the direction of the dipping needle, that is, the variation of the needle, at the place of construction. It is found that if the ship was built head north, the deviation is westerly when the ship's head is east, and easterly when the ship's head is west.

Induced magnetism is the magnetism induced in the soft iron in the vessel by the earth's magnetic force.

The earth itself is a huge magnet, whose magnetic poles are entirely distinct from its poles of rotation. The north magnetic pole is N.N.W. of Hudson Bay; the south magnetic pole is to the south of Tasmania. The magnetic equator crosses the geographical equator near the meridian of Greenwich and near 168° W. longitude, making with the geographical equator an angle of about 12°.

At all places on the magnetic equator a freely suspended needle takes a true horizontal position; that is, the dip of the needle is 0°. If the needle is taken to the north magnetic pole, the *red* or north end of the needle will point vertically downward; and at the south magnetic pole, vertically upward.

Retentive magnetism is the temporary magnetism induced in an iron ship when her head is kept in one direction for some time. Retentive magnetism frequently remains for days after the cause is removed.

Semicircular deviation is so named because it has the contrary name and maximum value in opposite semicircles; thus, if it is easterly on North, it is westerly on South.

It is caused by subpermanent magnetism and the transient induced magnetism of *vertical* masses of soft iron in the ship. In wooden ships it disappears when the ship's correct magnetic course is N. or S., and is greatest when the correct magnetic course is E. or W. In iron ships the neutral points (points of no semicircular deviation) are in the direction of the ship's head and stern while building. The part due to subpermanent magnetism remains the same in kind, either N. or S., but is different in amount in all latitudes, unless the ship is strained or subjected to mechanical violence. The part produced by induced magnetism in vertical iron changes from N. to S. in different latitudes with the inclination of the dipping needle. It disappears at the magnetic equator, and is of contrary names on opposite sides of this equator.

Quadrantal Deviation is caused by the transient induced magnetism of *horizontal* masses of soft iron in the ship. It disappears when the correct magnetic course is N., S., E., or W., and is greatest when the correct magnetic course is N.E., N.W., S.E., or S.W.; hence the name *quadrantal deviation.* It has the same name in opposite quadrants and contrary names in adjacent quadrants; thus, if it is easterly on N.E., it will be easterly on S.W., but westerly on S.E. and N.W. It remains the same in all magnetic latitudes and is not appreciably affected by the lapse of time.

The quadrantal deviation is compensated by soft-iron chains in boxes and horizontal cylinders of cast soft iron, or hollow cast-iron globes.

Retentive magnetism is corrected by over-compensating the quadrantal deviation when the course is E. and W., and by under-compensating when the course is N. and S.

The semicircular deviation caused by subpermanent magnetism is compensated by a system of magnets in the deck; that caused by induced magnetism by vertical soft-iron bars.

The amount of deviation remaining after the compass has been adjusted may be found as follows:

(1) *By the known bearing of a distant object on shore.* The ship is swung around so as to bring its head successively to the thirty-two points of the compass, and the bearing of the object on shore, as given by the standard compass when the ship's head is steadied on each point, is recorded in a table. The difference between the known correct magnetic bearing of the object and the successive bearings given by the standard compass gives the deviation for these points. A well-defined object should be selected so far distant that the diameter of the space through which the ship is swung will make no sensible difference in the real bearing from a central point. The Navy Department has established compass stations for convenience, with several points whose magnetic bearings are accurately determined.

(2) *By reciprocal bearings.* If there is no suitable object on land, a second compass may be placed on shore, and the mutual bearings of this instrument and the standard compass on shipboard, when the ship's head is steadied on a certain point, taken at the same instant by preconcerted signal. The bearing given by the compass on shore is then reversed. The difference between this reversed bearing and the bearing given by the standard compass is the deviation for this point of the compass. In the absence of cargoes containing iron, wooden vessels going short voyages have their deviation nearly constant, in which case it is a good plan to write it on each point of the compass for ready reference.

(3) *By amplitudes and azimuths of a heavenly body.* The deviation when at sea is determined from an azimuth. (See Sect. XXXI, p. 418.)

The deviation is named as follows: If the correct magnetic bearing of the distant object (or the reversed bearing of the shore compass) is to the *right* of the reading of the standard compass, the deviation is *easterly;* if the correct magnetic bearing of the distant object (or the reversed bearing of the shore compass) is to the *left* of the reading of the standard compass, the deviation is *westerly.*

A Deviation Table, or deviation card, is a tabulated statement of the amount of deviation for each point of the compass when the ship is on an even keel.

When the ship heels under sail or is rolling heavily there is an additional error introduced for various degrees of heeling, called the **heeling error,** which must be estimated and allowed for separately. Compensation is made by a vertical steel magnet placed exactly under the centre of the card when the vessel is upright.

Example 1. A ship is headed N.E. by standard compass. A distant object on shore bears S. 70° W. by the standard compass. The correct magnetic bearing of the object from the ship is known to be S. 89° W. Find the deviation.

Correct magnetic bearing	= S. 89° W., or 89° R. of S.
Observed bearing	= S. 70° W., or 70° R. of S.
Deviation (for the point N.E.)	= 19° E.

The deviation is east, because 89° R. of S. is to the right of 70° R. of S.

Example 2. A ship is headed E. by standard compass. The bearing of the compass on shore from the ship, as given by the standard compass, is S. 21° W. The standard compass in the ship bears N. 8° W. from the compass on shore. Find the deviation.

Correct magnetic bearing (shore reversed)	= S. 8° E.,	or 8° L. of S.
Observed bearing	= S. 21° W.,	or 21° R. of S.
Deviation (for the point E.)	=	29° W.

The deviation is west, because 8° L. of S. is to the left of 21° R. of S.

SOLUTION BY DIAGRAM

C.N. (Example 1) is the direction of the needle of the compass on ship.

N.E. is the direction in which the ship's head is pointed.

O is the direction of the object from the ship.

Draw a line so that the line *O* shall make with it an angle S. 89° W., and it will be the magnetic meridian.

Since the needle of the compass on ship points to the *right* of magnetic north, the deviation is *east*, and is evidently the angle formed by the two lines *M.N.* and *C.N.*

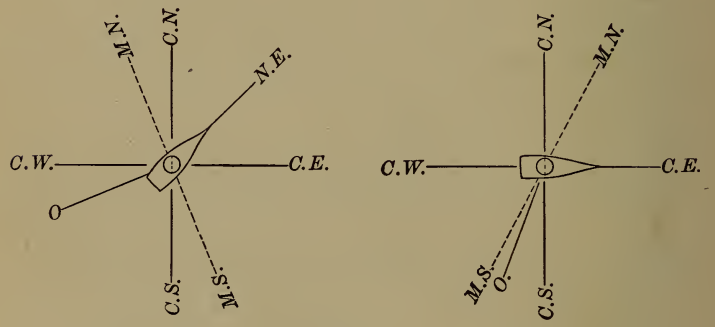


FIG. 2

C.N. (Example 2) is the direction of the needle of the compass on ship.

C.E. is the direction in which the ship's head is pointed.

O is the direction of the compass on shore from the ship.

As the magnetic bearing of the compass on ship from the compass on shore is N. 8° W., the magnetic bearing of the compass on shore from the compass on ship is S. 8° E.; hence, draw a line so that the line *O* shall make with it an angle S. 8° E., and it will be the magnetic meridian.

Since the needle of the compass on ship points to the *left* of magnetic north, the deviation is *west*, and is the angle formed by *M.N.* and *C.N.*

SECTION V

THE WIND

The wind is given the name of the point of the compass from which it blows.

Windward is the direction from which the wind blows.

Leeward is the direction toward which the wind blows.

When the wind is directly behind a ship it is said to be **right aft**, or **right astern**. The ship is then sixteen points from the wind.

A ship is said to be on the **starboard tack** or the **port tack** according as the wind blows on the *right* side or the *left* side of the ship.

When the wind blows at right angles to the side of a ship, the ship is said to have the wind **on her beam**. The ship is then eight points from the wind.

When a ship is more than eight points from the wind, the wind is said to be **abaft the beam**; when the ship is less than eight points from the wind, the wind is said to be **before the beam**.

A ship is said to be **close-hauled** when she is as near the wind as she will lie; that is, when she is headed as near as possible toward the point from which the wind blows. Few ships can lie closer than five points from the wind.

SECTION VI

LEEWAY

The angle which the fore-and-aft line of a vessel makes with her track is called **leeway**.

Let the arrow (Fig. 3) represent the direction of the wind, and *CB* the direction of the ship. Let *CB* represent the

distance the ship would go in a given time if she were not allowed to move sideways, and CD the distance she would go in the same time if she were not allowed headway. By a principle of mechanics, the ship would in the given time describe CA, the diagonal of the parallelogram of which CB and CD are adjacent sides.

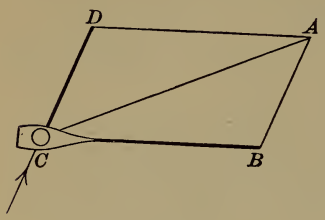


FIG. 3

The angle BCA is the leeway.

The amount of leeway depends upon the trim of the ship, her velocity through the water, the sails set, etc. It is sometimes found for a given ship, a given freight, and a given obliquity and velocity of the wind by special experiments.

In making up the reckoning, leeway and heave of the sea must be allowed for according to judgment, as they vary in different vessels according to their build, and in the same vessel according to her draught and trim for the time being. It depends also upon the amount of wind and sea and the sails carried, so that no fixed rule for estimating it can be laid down. The leeway table usually seen in books on Navigation is, therefore, of but little value.

SECTION VII

REDUCING THE COURSES

The **Compass Course** is the course steered.

The true course is equal to the compass course corrected for variation, deviation, and leeway.

From the preceding explanations, the following rules for changing from one course to the other are evident:

To find the true course from the compass course:

Variation { Allow easterly variation to the right.
Allow westerly variation to the left.

Deviation { Allow easterly deviation to the right.
Allow westerly deviation to the left.

Leeway { Allow leeway to the left on the starboard tack.
Allow leeway to the right on the port tack.

To find the compass course from the true course:

Variation { Allow easterly variation to the left.
Allow westerly variation to the right.

Deviation { Allow easterly deviation to the left.
Allow westerly deviation to the right.

Leeway { Allow leeway to the right on the starboard tack.
Allow leeway to the left on the port tack.

EXAMPLE. 1. The compass course is E.N.E., wind S.E., leeway 3 points, variation 10° E., deviation 2° 30′ E. Find the true course.

Compass course (E.N.E.) is 6 points	=	67° 30′ R. of N.
Leeway (starboard tack) is 3 points	=	33° 45′ L.
Compass course (corrected for leeway)	=	33° 45′ R. of N.
Variation and deviation (10° E. + 2° 30′ E.)	=	12° 30′ R.
True course	=	N. 46° 15′ E.

EXAMPLE 2. The compass course is W.S.W., wind S., leeway 2½ points, variation 30° E., deviation 18° W. Find the true course.

Compass course (W.S.W.) is 6 points	=	67° 30′ R. of S.
Leeway (port tack) is 2½ points	=	28° 7′ R.
Compass course (corrected for leeway)	=	95° 37′ R. of S.
	=	84° 23′ L. of N.
Variation and deviation (30° E. − 18° W. = 12° E.)	=	12° 0′ R.
True course	=	72° 23′ L. of N.
	=	N. 72° 23′ W.

EXAMPLE 3. The true course is N.E. by N., the vessel is on the port tack, leeway 10°, variation 1 point W., deviation 5° 30′ E. Find the compass course.

True course (N.E. by N.)	=	33° 45′ R. of N.
Variation is 1 point W.	=	11° 15′ R.
Correct magnetic course	=	45° 0′ R. of N.
Leeway (port tack)	=	10° 0′ L.
True course (corrected for variation and leeway)	=	35° 0′ R. of N.
Deviation (for 35° R. of N. = N.E. by N. nearly)	=	5° 30′ L.
Compass course	=	29° 30′ R. of N.
	=	N. 29° 30′ E.

SOLUTION BY DIAGRAM

The solutions of Examples 1, 2, and 3 are clearly shown by referring to Fig. 4.

T.N. is geographical north (True North).

M.N. is direction of needle by variation (Magnetic North).

C.N. is direction of needle by variation and deviation (Compass North).

C.C. is direction in which ship's head is pointed (Compass Course).

T.C. is direction in which ship goes owing to leeway (True Course).

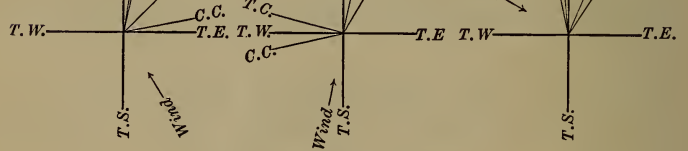


FIG. 4

Hence, in Example 1, the required course is evidently the angle formed by the two lines *T.N.* and *T.C.*

In Example 2, the required course is evidently the angle formed by the two lines *T.N.* and *T.C.*

In Example 3, the required course is evidently the angle formed by the two lines *C.N.* and *C.C.*

EXERCISE I

In the answers reckon 30″ or over as 1′ and reject less than 30″.

	Compass Course	Wind	Leeway	Variation	Deviation	True Course
1	S.	E.S.E.	1¼ pts.	52° 0′ W.	2° 0′ E.	Required
2	W.N.W.	N.	3 pts.	42° 0′ E.	18° 30′ W.	Required
3	S.S.E. ½ E.	S.W. ½ S.	3½ pts.	2¼ pts. E.	1¼ pts. W.	Required
4	Required	S. by W.	¾ pt.	10° 30′ E.	19° 0′ W.	S. 79° W.
5	W. ¼ N.	N.N.W.	1¾ pts.	8° 30′ E.	15° 35′ E.	Required
6	E. ¼ N.	N.N.E.	2¼ pts.	13° 0′ W.	20° 0′ E.	Required
7	Required	N. by W.	½ pt.	14° 0′ E.	19° 0′ E.	S. 85° E.
8	W.	N.N.W.	1¼ pts.	18° 30′ E.	21° 0′ W.	Required
9	E. ½ S.	N.N.E. ½ E.	2½ pts.	21° 0′ E.	4° 0′ W.	Required
10	Required	N. by W.	2¾ pts.	2 pts. W.	3½ pts. E.	E. by S. ¼ S.
11	Required	N.E.	3¼ pts.	2¾ pts. E.	1½ pts. E.	N. by W.
12	Required	S.S.W.	2½ pts.	2¾ pts. E.	¾ pt. E.	N.N.W.
13	Required	. . . . .	0 pts.	7° 0′ W.	15° 0′ W.	S. 64° E.
14	Required	. . . . .	0 pts.	6° 0′ E.	20° 0′ W.	N. 44° W.
15	N. 65° W.	. . . . .	0 pts.	10° 0′ E.	3° 0′ E.	Required
16	S. 15° W.	. . . . .	0 pts.	6° 0′ W.	18° 0′ E.	Required
17	S. 18° E.	. . . . .	0 pts.	25° 0′ E.	10° 0′ E.	Required
18	N. 30° E.	S. by W.	1¼ pts.	12° 0′ E.	10° 0′ W.	Required

SECTION VIII

THE LOG

The rate at which a vessel moves through the water is commonly determined by the **log.**

The **Common Log** consists of the log-chip (often called simply the *log*), the log line, and the sandglass.

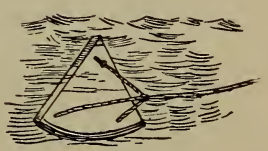


Fig. 5

The **Log-Chip** (Fig. 5) is a flat piece of wood in the form of

a sector, of about 5 inches radius, having its arc loaded with lead to make it float in an upright position, so that the vertex shall be just above the water.

The **Log Line**, to which the log-chip is attached by three strings, is an ordinary cord about 150 fathoms long. The *stray line* is the part of the log line next the log-chip, and is commonly about two thirds the length of the ship. The end of the stray line is marked by a rag tied into the line. The remainder of the line is divided into equal parts called *knots*, and each knot is divided into tenths. Beginning at the end of the stray line, the end of the first knot is marked by a piece of cord having one knot tied in it; the end of the second knot is marked by a cord with two knots tied in it; the end of the third knot is marked by a cord with three knots tied in it; and so on.

Fig. 6

The stray line allows the log to settle in the water clear of the ship before the measurement of the rate begins.

A **Sandglass** (Fig. 6) of 30 seconds, 28 seconds, or 14 seconds may be used.

The knot is made the same part of a nautical mile that the time required for the sand to run through the glass is of an hour.

To find the length of a nautical mile.

If the radius of the earth is 3960 statute miles, the circumference of the earth, $2\pi r$, is $2 \times 3.1416 \times 3960$ statute miles $= 24881.4720$ statute miles.

Length of $1° = 24881.4720$ statute miles $\div 360 = 69.1152$ statute miles.

Length of $1' = 1$ nautical mile $= \frac{69.1152}{60} \times 5280$ feet $=$ 6082 feet.

The knot corresponding to a 30-second glass would be $50\frac{2}{3}$ feet, determined by the following proportion, using 6080 feet as a nautical mile:

3600 seconds : 30 seconds = 6080 feet : $50\frac{2}{3}$ feet.

As a vessel generally overruns her reckoning, the knot corresponding to a 30-second glass is made 50 feet. The proportion would give $47\frac{1}{4}\frac{3}{5}$ feet as the length of the knot corresponding to a 28-second glass, but in practice it is made 46 feet 8 inches.

The log line is wound upon a *reel* (Fig. 7).

The operation of *heaving the log* is as follows: The officer throws the log-chip into the water on the lee side of the vessel. It remains nearly stationary in the water while the line is reeled off as the vessel moves forward. The instant that the mark indicating the end of the stray line passes over the side of the ship the glass is turned. The operator then counts the number of knots run out during the time required for the sand to run through the glass. The line is given a quick jerk, which detaches the two lower of the three cords by which the log-chip is attached to the line, the log falls flat, and is readily drawn in.

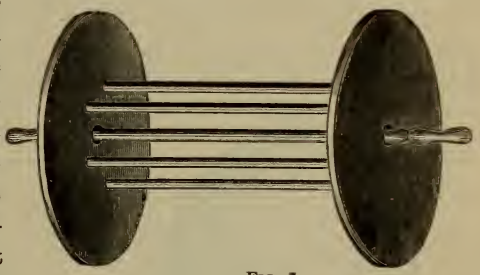


FIG. 7

Since each knot is (approximately) the same part of a geographical mile that the time is of one hour, the number of knots run out while the sand is running through the glass indicates the rate of the vessel in geographical miles per hour.

The **Patent Harpoon Log**, one form of which is shown in Fig. 8, consists of a brass cylinder, which is towed through the water after the vessel by a line about 50 fathoms long. The cylinder consists of two parts. The hinder part, or

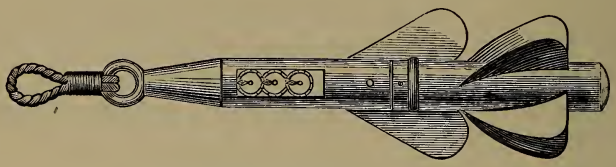


FIG. 8

rotator, is made to revolve by four curved blades as the log is drawn through the water; the forward part is kept from revolving by two straight blades, and contains a registering apparatus connected with the rotator, which shows the distance.

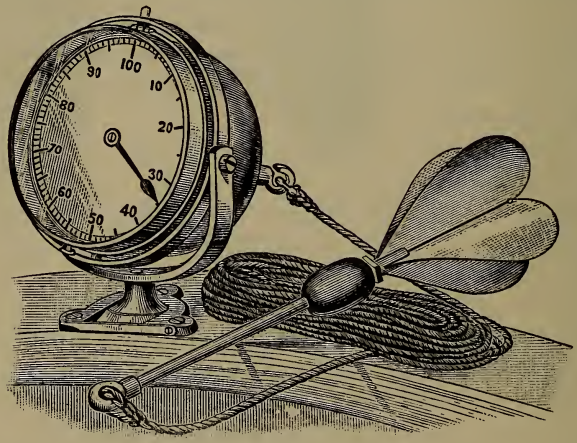


FIG. 9

The **Patent Taffrail Log**, one form of which is shown in Fig. 9, consists of a registering apparatus which is permanently attached to the taffrail of the vessel, and a rotator

which is towed through the water by a cord connected with the register. The revolutions of the rotator are communicated to the register by the cord.

The **Ground Log** consists of the ordinary log line with a hand lead substituted for the log-chip. When it is hove, the lead sinks to the bottom and remains stationary. The line reeled off gives the speed *over the bottom.* This log is not affected by tides or currents. It may be used in shallow water.

The rate of paddle steamers can be estimated from the number of revolutions of the paddle wheels quite as accurately as by the best patent log, especially in smooth water.

The Different Logs compared

The best results are obtained by the use of the harpoon log, *if it is in good order;* but the rapid motion of parts of the registering apparatus soon puts the best patent log out of order. Hence, fast ocean steamers ordinarily use the common log (hove every two hours) when in the open sea, and both the common and the patent log when nearing land.

The taffrail log is the most convenient, since the registering apparatus can be examined at any time without taking the rotator from the water; and if the rotator is lost, it may be replaced at small cost, whereas if the harpoon log is lost (a not infrequent occurrence), the cost of replacing it is considerable. Although the taffrail log is growing in favor, it is considered not as trustworthy as the harpoon log.

Example 1. What should be the length of the knot on the log line when the sandglass runs out in 27 seconds, the nautical mile being 6080 feet? *Ans.* $45\frac{3}{5}$ feet.

Example 2. A log line is knotted every $59\frac{1}{3}$ feet. Find what glass must be employed, the nautical mile being 6080 feet. *Ans.* 35 seconds.

SECTION IX

CURRENTS

The *set* of a current is its direction, or the point of the compass toward which it is moving.

The *drift* of a current is its rate in miles per hour.

EXAMPLE 1. If a log is thrown from a ship in a current, will it show the rate of sailing or the rate over the bottom?

Ans. In a current, the log gives the speed through the water and not the speed over the bottom, since the log as well as the ship is carried by the current.

The set and drift of a current may be ascertained by taking, at a short distance from the ship, a boat which is kept from being carried by the current by letting down to a considerable depth a heavy weight attached by a rope to the stern of the boat. The method of allowing for the effect of a current upon a ship will be explained hereafter.

EXAMPLE 2. The wind is carrying a ship at the rate of 15 miles an hour; a current gives her a lateral velocity of 10 miles an hour. Find the velocity of the ship over the

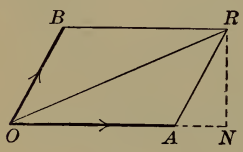


FIG. 10

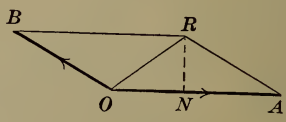


FIG. 11

bottom when the direction in which the current carries the ship makes (1) an angle of 60° and (2) an angle of 150° with the direction in which the wind is carrying the ship. (See Sect. VI, p. 316.)

Let *OA* (Figs. 10 and 11) represent the distance and direction the ship would be carried by the wind in one hour if

there was no current, and let OB represent the distance and direction the ship would be carried by the current in one hour if there was no wind. Then OR represents the distance and direction the ship is carried in one hour by the wind and current acting together.

$\overline{OR}^2 = (OA + AN)^2 + \overline{NR}^2$.

$AN = AR\cos 60° = 10 \times \frac{1}{2} = 5$.

$NR = AR\sin 60° = 10 \times \frac{1}{2}\sqrt{3} = 5\sqrt{3}$.

$\overline{OR}^2 = (15 + 5)^2 + (5\sqrt{3})^2 = 475$.

$OR = \sqrt{475} = 21.8$.

Therefore, the velocity of the ship over the bottom is 21.8 miles an hour.

$\overline{OR}^2 = (OA - NA)^2 + \overline{NR}^2$.

$NA = AR\cos 30° = 10 \times \frac{1}{2}\sqrt{3} = 5\sqrt{3}$

$NR = AR\sin 30° = 10 \times \frac{1}{2} = 5$.

$\overline{OR}^2 = (15 - 5\sqrt{3})^2 + 5^2$

$= 325 - 150\sqrt{3}$.

$OR = \sqrt{325 - 150\sqrt{3}} = 8.07$.

Therefore, the velocity of the ship over the bottom is 8.07 miles an hour.

CHAPTER II

THE SAILINGS

SECTION X

DEFINITIONS

The **Log Book** is a record of the ship's course, containing the data from which the navigator establishes the ship's position by dead reckoning.

The **Difference of Latitude** of two places is the arc of a meridian included between the parallels of the two places.

The **Middle Latitude** of two places is the latitude of a parallel midway between the two places.

The **Difference of Longitude** of two places is the arc of the equator included between the meridians of the two places.

The **Departure** is the distance measured on a parallel between the meridian left and the meridian reached.

SECTION XI

NOTATION

C = the true course.
D = the distance in geographical miles.
L = latitude in general.
L' = the latitude left (latitude from).
L'' = the latitude arrived at (latitude in).
L_m = the middle latitude.
L_d = the difference of latitude.

λ = longitude in general.
λ' = the longitude left (longitude from).
λ'' = the longitude arrived at (longitude in).
λ_d = the difference of longitude.
p = the departure.

SECTION XII

PLANE SAILING

Given the course and the distance, to find the difference of latitude and the departure:

Case I

When the distance is so small that the curvature of the earth may be neglected.

Let CA (Fig. 12) be the distance sailed, CB the meridian through C, and AB the parallel through A.

Then, angle $ACB = C$,
$CA = D$,
$CB = L_d$,
$AB = p$.

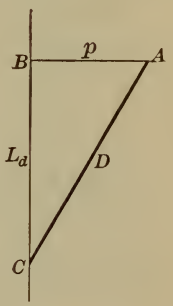


Fig. 12

By Plane Trigonometry,

$$\sin C = \frac{p}{D}, \quad \cos C = \frac{L_d}{D}, \quad \tan C = \frac{p}{L_d};$$

$$p = D \sin C, \quad L_d = D \cos C, \quad p = L_d \tan C;$$

$$D = \frac{p}{\sin C}, \quad D = \frac{L_d}{\cos C}, \quad L_d = \frac{p}{\tan C}.$$

Case II

When the distance is so great that the curvature of the earth cannot be neglected.

Let PCD (Fig. 13) represent a portion of the earth's surface, P the pole, CA the ship's track between C and A, PC and PD the meridians through C and A, CD and BA the parallels through C and A.

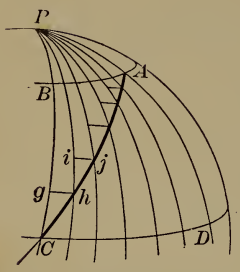


FIG. 13

Let CA be divided into parts Ch, hj, $\cdots$, so small that the curvature of the earth may be neglected for these distances; and let gh, ij, $\cdots$, be intercepts on the parallels corresponding to Ch, hj, $\cdots$

$$\text{Angle } PCA = PhA = PjA = \cdots = C.$$

By Plane Trigonometry,

$$gh = Ch \sin C,$$
$$ij = hj \sin C,$$
$$\cdots = \cdots$$
$$\therefore gh + ij + \cdots = (Ch + hj + \cdots) \sin C.$$

Hence, $$p = D \sin C. \qquad \text{(a)}$$

Again, $$Cg = Ch \cos C,$$
$$hi = hj \cos C,$$
$$\cdots = \cdots$$
$$\therefore Cg + hi + \cdots = (Ch + hj + \cdots) \cos C.$$

Hence, $$L_d = D \cos C. \qquad \text{(b)}$$

From (a) and (b),

$$\left.\begin{array}{l} \sin C = \dfrac{p}{D}, \\ p = D \sin C, \\ D = \dfrac{p}{\sin C}. \end{array}\right\} [1] \qquad \left.\begin{array}{l} \cos C = \dfrac{L_d}{D}, \\ L_d = D \cos C, \\ D = \dfrac{L_d}{\cos C}. \end{array}\right\} [2] \qquad \left.\begin{array}{l} \tan C = \dfrac{p}{L_d}, \\ p = L_d \tan C, \\ L_d = \dfrac{p}{\tan C}. \end{array}\right\} [3]$$

These formulas are the same as in Case I; but in Case II p is the sum of the partial departures, each being taken in the

latitude of its triangle, and is evidently greater than BA and less than CD.

Plane Sailing, as here treated, may be defined as the method of establishing the relations between course, distance, difference of latitude, and departure, on the supposition that small portions of the earth's surface may be regarded as plane.

The several problems are readily solved by formulas [1], [2], and [3], but the method by inspection is commonly used by navigators. A number of examples will be solved by this method in order to illustrate the use of the Traverse Tables.

EXAMPLE 1. A ship sails from latitude 40° 20′ N. on a N.N.E. course for 92 miles. Find the departure and the latitude in.

SOLUTION I. *By computation.* The true course N.N.E. = N. 22° 30′ E.

To find the departure:

By [1], $p = D \sin C$.

$D = 92$ | $\log D = 1.96379$
$C = 22° 30′$ | $\log \sin C = \underline{9.58284}$
$\log p = \underline{1.54663}$
$\therefore p = 35.2$ miles.

To find the difference of latitude:

By [2], $L_d = D \cos C$.

$\log D = 1.96379$
$\log \cos C = \underline{9.96562}$
$\log L_d = \underline{1.92941}$
$\therefore L_d = 85.0$ miles
$= 1° 25′$ N.
$L' = 40° 20′$ N.
$\therefore L'' = 41° 45′$ N.

SOLUTION II. *By inspection of the Traverse Table.* Since the hypotenuse and an acute angle of a right triangle are given to find the sides, the Traverse Table may be used. Corresponding to the course N.N.E. (2 points), and distance 92 miles, Table VIII* gives latitude = 85 and departure = 35.2. The difference of latitude, 85 miles, is 1° 25′.

EXAMPLE 2. A ship sails S.W. by W. 488 miles. Find the difference of latitude and the departure.

By inspection of the Traverse Table. S.W. by W. = 5 points. In Table VIII, at the bottom of page 87, we find 5 points; hence, we look at the bottom of the page for the designations of the columns. The table

* Wentworth and Hill's Logarithmic and Trigonometric Tables.

extends only to a distance of 300; hence, we find the latitudes and departures corresponding to the distances 300 and 188, and take their sum.

For distance 300	latitude = 166.7	departure = 249.4
" " 188	" = 104.4	" = 156.3
" " 488	" = 271.1	" = 405.7

Or, we may divide the given distance by any convenient number, and find the latitude and departure corresponding to the quotient (the course remaining unchanged), then multiply each of these by the divisor of the distance.

$488 \div 2 = 244.$

For distance 244	latitude = 135.6	departure = 202.9
2	2	2
For distance 488	latitude = 271.2	departure = 405.8

The results differ slightly by the two methods; but this small discrepancy is of no importance in practice. (See Sect. XVIII, p. 366.)

EXAMPLE 3. Course $6\frac{1}{2}$ points, distance 21.7; required the difference of latitude and the departure.

By inspection of the Traverse Table. $6\frac{1}{2}$ points is found at the bottom of p. 81, Table VIII. Corresponding to distance 217 (21.7×10) the difference of latitude is 63.0, and the departure 207.7. Dividing each of these numbers by 10, the required difference of latitude is found to be 6.3, and the departure 20.8.

EXAMPLE 4. If a ship runs S.E. by S. from 1° 44′ north latitude, and is then by observation in 2° 46′ south latitude, what are the distance and the departure?

S.E. by S. = 3 points.

1° 44′ N.
2° 46′ S.
4° 30′ = 270 miles = difference of latitude.

On p. 87, Table VIII, we find 3 points at the top; but the greatest difference of latitude on this page is 249; hence, we divide the given difference of latitude by 2. $270 \div 2 = 135$. The nearest difference of latitude to 135 is 134.7, opposite which the distance is 162 and the departure 90. Multiplying each of these numbers by 2, the required distance is found to be 324 and the departure 180.

EXAMPLE 5. A ship sails N. 25° W. until the departure is 98 miles. Find the distance and the difference of latitude.

SOLUTION I. *By computation.*

To find the distance:

By [1], $D = p \csc C$.

$C = 25°$, $p = 98$

$\log p = 1.99123$
$\log \csc C = \underline{0.37405}$
$\log D = 2.36528$
$\therefore D = 231.9.$

To find the difference of latitude:

By [3], $L_d = p \cot C$.

$\log p = 1.99123$
$\log \cot C = \underline{0.33133}$
$\log L_d = 2.32256$
$\therefore L_d = 210.2.$

SOLUTION II. *By inspection of the Traverse Table.* Since the course is given in *degrees*, we turn to Table IX. At the top of p. 116 we find 25°, and opposite departure 98 we find the required distance 232, and the difference of latitude 210.3.

EXAMPLE 6. Find the course and the distance corresponding to a difference of latitude of 696 miles and a departure of 186 miles, the course lying between N. and E.

SOLUTION I. *By computation.*

To find the course:

By [3], $\tan C = \dfrac{p}{L_d}$.

$p = 186$, $L_d = 696$

$\log p = 2.26951$
$\operatorname{colog} L_d = \underline{7.15739 - 10}$
$\log \tan C = 9.42690$
$\therefore C = \text{N. } 14° 58' \text{ E.}$

To find the distance:

By [1], $D = \dfrac{p}{\sin C}$.

$\log p = 2.26951$
$\operatorname{colog} \sin C = \underline{0.58795}$
$\log D = 2.85746$
$\therefore D = 720.2.$

SOLUTION II. *By inspection of the Traverse Table.* Since the given difference of latitude exceeds the limits of the tables, we divide each of the given numbers by 3: $696 \div 3 = 232$; $186 \div 3 = 62$. If the course is desired in degrees, we now turn to Table IX and look for a difference of latitude of 232 opposite a departure of 62. On p. 106 the nearest approximation is found. At the top of this page the angle is 15°; hence, the required course is N. 15° E. If the course is desired in points, we turn to Table VIII. The nearest approximation is found on p. 80, at the top of which stands $1\frac{1}{4}$ points. Hence, the required course is N. by E. $\frac{1}{4}$ E. The distance is 720 ($= 240 \times 3$) miles.

NOTE. The answers given at the end of this volume are obtained by computation. The student should also solve by inspection.

EXERCISE II

	L'	L''	C	D	p
1	49° 57′ N.	Required	S.W. by W.	488.0	Required
2	1° 45′ N.	Required	S.E. by E.	487.8	Required
3	3° 15′ S.	Required	N.E. by E. $\frac{3}{4}$ E.	449.1	Required
4	2° 10′ S.	Required	N. by E.	267.0	Required
5	41° 30′ N.	Required	S.S.W.	295.5	Required
6	21° 59′ S.	24° 49′ S.	Required	360.0	Required
7	2° 9′ S.	3° 11′ N.	Required	354.0	Required
8	1° 30′ N.	0° 26′ S.	S. by W.	Required	Required
9	40° 17′ N.	37° 6′ N.	S. by W. $\frac{1}{2}$ W.	Required	Required
10	38° 0′ N.	Required	S.W. by W.	Required	48.2
11	18° 25′ N.	Required	S.W. by W. $\frac{3}{4}$ W.	Required	65.1
12	50° 18′ N.	54° 48′ N.	Required	299.0	Required
13	32° 30′ N.	19° 59′ N.	Required	812.0	Required
14	2° 8′ S.	Required	N. 11° E.	500.0	Required
15	20° 21′ S.	Required	N. 20° E.	402.0	Required
16	40° 25′ S.	Required	N. 87° E.	240.0	Required
17	20° 48′ N.	17° 13′ N.	Required	Required	289.2 W.
18	51° 45′ N.	53° 11′ N.	Required	Required	128.0 E.
19	0° 20′ S.	0° 18′ N.	Required	Required	142.7 E.
20	40° 20′ N.	41° 37′ N.	Required	Required	52.6 W.

SECTION XIII

PARALLEL SAILING

Parallel Sailing is that case of sailing in which the track is on a parallel of latitude.

Let P (Fig 15) represent the pole, O the centre of the earth, A and B two places on the same parallel AB, PAE and PBQ meridians through A and B intercepting EQ on the equator. Let AD and BD be perpendicular to PO. Draw AO. AB is the departure, or meridian distance Regard the earth as spherical.

$AE = BQ = EOA = OAD =$ latitude of A and B.

In the right triangle AOD,

$$\cos ODA = \frac{AD}{AO} = \frac{AD}{R}, \text{ or } \cos L = \frac{AD}{R}.$$

$$\therefore AD = R \cos L.$$

By Geometry, $EQ : AB = EO : AD$,

or $$\lambda_d : p = R : R \cos L.$$

$$\therefore \mathbf{p = \lambda_d \cos L},$$

and $$\mathbf{\lambda_d = p \sec L} \quad [4]$$

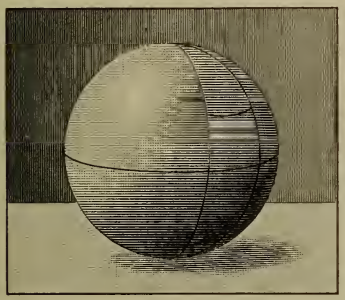


FIG. 14

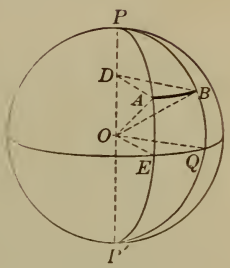


FIG. 15

These relations may be graphically represented by a right plane triangle.

Now $p = \lambda_d \cos L$. Therefore, λ_d is the hypotenuse of the right triangle of which L is an acute angle and p the leg adjacent, as in Fig. 16.

Since $p = \lambda_d \cos L$, the meridian distance varies as the cosine of the latitude.

Let $\lambda_d = 1° = 60' = 60$ miles; then the above formula becomes $p = 60 \cos L$.

Hence, to find the number of geographical miles in 1° of longitude at any place, multiply 60 by the cosine of the latitude.

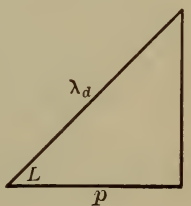


FIG. 16

The number of statute miles in a degree of longitude may be found by multiplying the numbers in the table by $\frac{69.115}{60}$, or by using the formula $p = 69.115 \cos L$.

TABLE SHOWING THE NUMBER OF GEOGRAPHICAL MILES IN A DEGREE OF LONGITUDE FOR EACH DEGREE OF LATITUDE

L	MILES	L	MILES	L	MILES	L	MILES	L	MILES	L	MILES
1°	59.99	16°	57.67	31°	51.43	46°	41.68	61°	29.09	76°	14.51
2°	59.96	17°	57.38	32°	50.88	47°	40.92	62°	28.17	77°	13.50
3°	59.92	18°	57.06	33°	50.32	48°	40.15	63°	27.24	78°	12.48
4°	59.85	19°	56.73	34°	49.74	49°	39.36	64°	26.30	79°	11.45
5°	59.77	20°	56.38	35°	49.15	50°	38.57	65°	25.36	80°	10.42
6°	59.67	21°	56.01	36°	48.54	51°	37.76	66°	24.41	81°	9.38
7°	59.56	22°	55.63	37°	47.92	52°	36.94	67°	23.44	82°	8.35
8°	59.42	23°	55.23	38°	47.28	53°	36.11	68°	22.48	83°	7.32
9°	59.26	24°	54.81	39°	46.63	54°	35.27	69°	21.50	84°	6.28
10°	59.09	25°	54.38	40°	45.96	55°	34.41	70°	20.52	85°	5.23
11°	58.89	26°	53.93	41°	45.28	56°	33.55	71°	19.53	86°	4.18
12°	58.69	27°	53.46	42°	44.59	57°	32.68	72°	18.54	87°	3.14
13°	58.46	28°	52.97	43°	43.88	58°	31.79	73°	17.54	88°	2.09
14°	58.22	29°	52.47	44°	43.16	59°	30.90	74°	16.54	89°	1.05
15°	57.95	30°	51.96	45°	42.43	60°	30.00	75°	15.53	90°	0.00

EXAMPLE 1. A ship in latitude 42° changes her longitude 3° 20′ by sailing on this parallel. Find her departure.

SOLUTION I

By [4], $p = \lambda_d \cos L$.

$p = 60 \times 3\frac{1}{3} \times \cos 42°$

$= 60 \times 3\frac{1}{3} \times 0.7431$

$= 148.6$ miles.

SOLUTION II

The table gives for $L = 42°$,

1° of $\lambda = 44.59$.

$\therefore 3\frac{1}{3}°$ of $\lambda = 3\frac{1}{3} \times 44.59$

$= 148.6$ miles.

SOLUTION III. By referring to Fig. 16, it will be seen that the given latitude corresponds to course, and difference of longitude to distance. Hence, p is the latitude in the Traverse Table corresponding to course 42° and distance $60 \times 3\frac{1}{3} = 200$, which gives 148.6 miles, as before.

EXAMPLE 2. A ship sails 176.2 miles due west from the Lizard, in latitude 49° 57′ N. and longitude 5° 14′ W. Find the longitude in.

SOLUTION I

By [4], $\lambda_d = p \sec L$.

$p = 176.2$	$\log p = 2.24601$
$L = 49° 57'$	$\log \sec L = 0.19148$
$\lambda' = 5° 14'$ W.	$\log \lambda_d = 2.43749$
$\lambda_d = 4° 34'$ W.	$\therefore \lambda_d = 273.8$ miles
$\lambda'' = 9° 48'$ W.	$= 4° 34'$ W.

SOLUTION II

From the table,

$1° \lambda$ for $49° L = 39.36$,

$1° \lambda$ for $50° L = 38.57$.

By interpolation,

$1° \lambda$ for $49° 57' L = 38.61$.

$176.2 \div 38.61 = 4.56$.

$\lambda_d = 4.56° = 4° 34'$.

EXERCISE III

	L	p	λ'	λ
1	55° 55′	Required	2° 10′ W.	12° 52′ E.
2	52° 00′	Required	0° 59′ W.	2° 24′ E.
3	61° 25′	Required	179° 20′ W.	176° 52′ E.
4	56° 00′	Required	3° 12′ W.	4° 8′ E.
5	80° 00′	Required	10° 0′ W.	17° 41′ W.
6	60° 00′	204.0 E.	160° 2′ E.	Required
7	51° 28′	70.9 E.	32° 7′ W.	Required
8	64° 16′	265.7 W.	170° 0′ W.	Required
9	46° 37′	352.0 E.	163° 42′ E.	Required
10	39° 57′	398.0 W.	4° 8′ W.	Required

11. From latitude 32° 3′ S., longitude 179° 45′ W., a ship makes 54 miles west (true). Required the longitude in.

12. From latitude 35° 30′ S., longitude 27° 28′ W., a ship sails east (true) 301 miles. Required the longitude in and the compass course; variation $1\frac{3}{4}$ points E., leeway $\frac{1}{4}$ point, the ship being on the port tack, deviation 8° 50′ E.

NOTE. In changing from the true course to the compass course, apply variation and leeway, then deviation, which is supposed to be given for the course to which it is directly applied.

SECTION XIV

MIDDLE LATITUDE SAILING

Middle Latitude Sailing is a combination of Plane and Parallel Sailings, chiefly for the purpose of determining difference of longitude. It is based on the supposition that the departure in Plane Sailing is equal to the distance between the meridians passing through the extreme points of the rhumb line, measured on the middle parallels between these points.

Let CA (Fig. 18) represent the ship's track, CD and BA parallels, and PBC and PAD meridians through C and A, respectively. Let EF be the parallel midway between C and A.

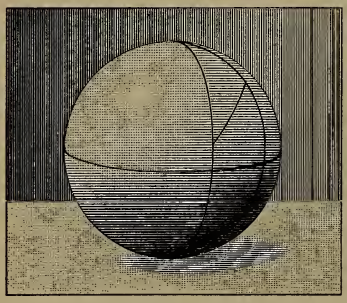


Fig. 17

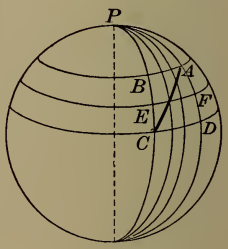


Fig. 18

It is shown in Plane Sailing that p is less than CD and greater than BA. It is nearly equal to EF, and exactly equal to an intercept a little nearer to the pole than EF, for it is plain that the middle latitude distance receives a much greater accession than the departure if the track CA cuts the successive meridians at a very small angle.

Hence, $L_m = \frac{1}{2}(L' + L'')$, nearly, if both places are on the same side of the equator.

In Middle Latitude Sailing p is first found as in Plane Sailing, then λ_d is found, on the supposition that λ_d is equal to

the change of longitude which the ship would have made by sailing a distance p on a parallel in latitude L_m.

If we substitute for p its value in the formulas,

In Parallel Sailing,

By [4], $\mathbf{p = \lambda_d \cos L_m}$, $\mathbf{\lambda_d = p \sec L_m}$. [5]

In Plane Sailing,

By [1] and [3], $p = D \sin C$, $\tan C = \frac{p}{L_d}$.

Hence, by combining,

$$\mathbf{\tan C = \frac{\lambda_d \cos L_m}{L_d}}; \quad [6]$$

$$\mathbf{\lambda_d = D \sin C \sec L_m}. \quad [7]$$

The formula $L_m = \frac{1}{2}(L' + L'')$ may be used without sensible error in low latitudes when the course is not less than about 45° and both places are on the same side of the equator.

When the places are on different sides of the equator and the distance not great, $p = \lambda_d$ may be used, since the meridians near the equator are sensibly parallel; but when the distance is considerable the two portions of the track on different sides of the equator should be treated separately.

Let CA (Fig. 19) represent the rhumb line crossing the equator EQ at B. Let CE and AQ represent meridians.

For the rhumb CB the middle latitude is $\frac{1}{2}$ CE; hence, EB (the difference of longitude of C and B) may be found. For the rhumb BA the middle latitude is $\frac{1}{2}$ AQ; hence, BQ (the difference of longitude of B and A) may be found.

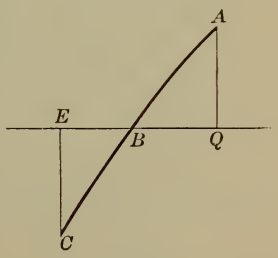


FIG. 19

Then, $\lambda_d = EB + BQ$.

In high latitudes, or in any latitude, if the course is less than about 45° and the distance great,

$$L_m = \tfrac{1}{2}(L' + L'') + d,$$

d being a small arc. By means of a small table of corrections, the imperfections of the middle latitude method may be removed and the results rendered accurate. This correction of the middle latitude is seldom used in practice.

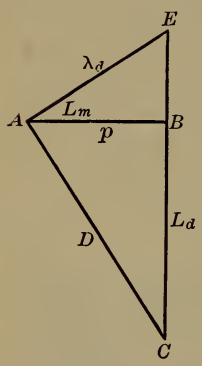


FIG. 20

The relations of the elements of Middle Latitude Sailing may be graphically represented by combining the triangles of Plane and Parallel Sailings, as in Fig. 20.

$$ACB = C, \qquad AB = p,$$
$$CA = D, \qquad BAE = L_m,$$
$$CB = L_d, \qquad AE = \lambda_d.$$

EXAMPLE 1. A ship sails from Sandy Hook light, in latitude 40° 28′ N., longitude 74° W., on an E.S.E. course, 62 miles. Find the latitude in and the longitude in.

SOLUTION 1. *By computation.*

To find the latitude in:

By [2], $L_d = D \cos C$.

$D = 62$ | $\log D = 1.79239$
$C = 67° 30'$ | $\log \cos C = \underline{9.58284}$
$\log L_d = 1.37523$

$\therefore L_d = 23.7' = 24'$ S.

$L' = 40° 28'$ N.
$L_d = \underline{\quad 24'}$ S.
$L'' = 40° \ 4'$ N.

To find the middle latitude:

$L' = 40° 28'$ N.
$L'' = 40° \ 4'$ N.
$2 \overline{)\, 80° 32'}$
$L_m = 40° 16'$ N.

To find the longitude in:

By [7], $\lambda_d = D \sin C \sec L_m$.

$\log D = 1.79239$
$\log \sin C = 9.96562$
$\log \sec L_m = \underline{0.11745}$
$\log \lambda_d = 1.87546$

$\therefore \lambda_d = 75.1'$
$= 1° 15'$ E.

$\lambda' = 74° \ 0'$ W.
$\lambda_d = \underline{1° 15'}$ E.
$\lambda'' = 72° 45'$ W.

Solution II. *By inspection.* In Fig. 20 it is evident that if we enter the Traverse Table with the course and distance, the difference of latitude and departure may be found as in Plane Sailing, p. 329, Example 1. Corresponding to the course E.S.E. and distance 62, Table VIII, p. 83, gives the difference of latitude 23.7 and the departure 57.3; whence, the latitude in and middle latitude may be found as in Solution I.

In Fig. 20 the difference of longitude AE is the distance corresponding to the course E and the departure AB. But the angle $E = 90° - BAE$; that is, the angle E is the complement of the middle latitude. In this example the middle latitude is 40° 16′; whence, the complement of the middle latitude is $90° - 40° 16′ = 49° 44′ = 50°$, approximately. Corresponding to the course 50° and the departure 57.3 (the nearest is 57.5), Table IX, p. 131, gives the distance 75, which is the difference of longitude required; whence, the longitude in may be found as in Solution I.

Example 2. A ship sails from latitude 20° N., longitude 160° W., until the difference of latitude is 1° 50′ S. and the departure 440.2 miles W. Find by inspection the latitude in, the longitude in, the course, and the distance.

$L' = 20°\ 0'$ N.	$L' = 20°\ 0'$ N.	$\lambda' = 160°\ 0'$ W.
$L_d = 1°\ 50'$ S.	$L'' = 18°\ 10'$ N.	$\lambda_d = 7°\ 45'$ W.
$L'' = 18°\ 10'$ N.	$2)38°\ 10'$	$\lambda'' = 167°\ 45'$ W.
	$L_m = 19°\ 5'$ N.	
	co. $L_m = 70°\ 55'$	

To find the longitude in. Since 440.2 exceeds the limits of the table, one third of this number is taken. $440.2 \div 3 = 146.7$. Corresponding to course 71° (= co. L_m) and departure 146.7, Table IX, p. 110, gives the distance 155. $3 \times 155 = 465 = 7° 45'$, which is the difference of longitude; whence, the longitude in is found to be 167° 45′ W.

To find the course and the distance. Since the departure 440.2 exceeds the difference of latitude 110 (= 1° 50′), the course and designation of the columns must be sought at the bottom of the page. $110 \div 2 = 55$, and $440.2 \div 2 = 220.1$. Corresponding to the difference of latitude 55 and the departure 220.1 (the nearest are 54.9 and 220.3), Table IX, p. 105, gives the distance 227. $2 \times 227 = 454$. At the bottom of the page the course 76° is found. Hence, the course and the distance required are S. 76° W. and 454 miles.

Example 3. A navigator wishes to sail from the Lizard, in latitude 49° 57′ N., longitude 5° 14′ W., to St. Mary's Island,

in latitude 37° N. and longitude 25° 6′ W. Find the true course and the distance.

To find L_d:

$L' = 49° 57'$ N.
$L'' = 37° 0'$ N.
$L_d = 12° 57'$ S.
$= 777$ miles.

To find L_m:

$L' = 49° 57'$ N.
$L'' = 37° 0'$ N.
$2)86° 57'$
$L_m = 43° 29'$ N.

To find λ_d:

$\lambda' = 5° 14'$ W.
$\lambda'' = 25° 6'$ W.
$\lambda_d = 19° 52'$ W.
$= 1192$ miles.

To find the course:

By [6], $\tan C = \dfrac{\lambda_d \cos L_m}{L_d}$.

$\lambda_d = 1192$
$L_m = 43° 29'$
$L_d = 777$

$\log \lambda_d = 3.07628$
$\log \cos L_m = 9.86068$
$\operatorname{colog} L_d = 7.10958 - 10$
$\log \tan C = 10.04654$

$\therefore C =$ S. 48° 4′ W.

To find the distance:

By [2], $D = L_d \sec C$

$\log L_d = 2.89042$
$\log \sec C = 0.17505$
$\log D = 3.06547$

$\therefore D = 1163$ miles.

EXERCISE IV

	L'	L''	λ'	λ''	C	D
1	25° 35′ N.	27° 28′ N.	60° 0′ W.	54° 55′ W.	Required	Required
2	32° 30′ N.	34° 10′ N.	25° 24′ W.	29° 8′ W.	Required	Required
3	39° 30′ S.	41° 0′ S.	74° 20′ E.	70° 12′ E.	Required	Required
4	46° 24′ S.	Required	178° 28′ E.	Required	S.E. $\frac{3}{4}$ E.	278.0
5	20° 29′ N.	Required	179° 10′ W.	Required	W. by S. $\frac{1}{2}$ S.	333.0
6	0° 56′ N.	Required	29° 50′ W.	Required	S. 47° E.	168.0
7	42° 25′ N.	Required	66° 14′ W.	Required	S.E. by E.	25.0
8	42° 8′ N.	Required	65° 48′ W.	Required	E. $\frac{1}{2}$ S.	126.0
9	41° 52′ N.	Required	62° 47′ W.	Required	E. $\frac{1}{2}$ S.	161.0
10	41° 38′ N.	41° 26′ N.	59° 16′ W.	Required	E. by S.	Required
11	41° 19′ N.	41° 11′ N.	57° 47′ W.	Required	Required	167.0
12	46° 28′ N.	45° 17′ N.	22° 18′ W.	19° 39′ W.	Required	Required
13	25° 30′ S.	28° 15′ S.	2° 15′ E.	11° 17′ E.	Required	Required
14	33° 40′ N.	30° 49′ N.	13° 20′ E.	17° 56′ E.	Required	Required
15	19° 30′ S.	17° 24′ S.	0° 10′ E.	1° 28′ W.	Required	Required

16. A ship sails from Boston lighthouse, in latitude 42° 20′ N., longitude 71° 4′ W., on a N.N.E. course, 184 miles. Find the latitude in and the longitude in.

17. A ship sails from Cape May, in latitude 38° 56′ N., longitude 74° 57′ W., on a S.S.E. course, 240 miles. Find the latitude in and the longitude in.

18. A ship sails from Cape Cod light, in latitude 42° 2′ N., longitude 70° 3′ W., on an E. by N. compass course, 170 miles; wind S.E. by S., leeway $\frac{1}{4}$ point, deviation $17\frac{3}{4}°$ E., variation $11\frac{1}{4}°$ W. Find the latitude in and the longitude in.

19. A ship sails from Cape Cod light, on a S.S.E. compass course, 140 miles; deviation $5\frac{1}{2}°$ E., variation $11\frac{1}{4}°$ W. Find the latitude in and the longitude in.

20. A ship sails from latitude 55° 1′ N., longitude 1° 25′ W., on a S.W. compass course, 101 miles; wind W.N.W., leeway $1\frac{1}{4}$ points, deviation 6° W., variation 24° 56′ W. Find the latitude in and the longitude in.

21. A ship sails from the Bermudas, in latitude 32° 18′ N., longitude 64° 50′ W., on a W.S.W. compass course, 190 miles; deviation 1 point W., variation 1 point W. Find the latitude in and the longitude in.

22. A ship sails from the Bermudas, on a W.N.W. compass course, 90 miles; wind S.W., leeway 1 point, deviation 1 point E., variation 1 point W. Find the latitude in and the longitude in.

23. A navigator wishes to sail on a rhumb from the Bermudas to Cape Fear, in latitude 33° 52′ N., longitude 78° W.; variation 10° W., deviation 7° W. Find the compass course and the distance.

24. A ship from latitude 36° 32′ N. sails between south and west until she has made 480 miles of departure and 9° 22′ of difference of longitude. Find the latitude in, the course, and the distance. (Take $L_m = \frac{1}{2}(L' + L'') + 13'$.)

SECTION XV

MERCATOR'S SAILING

Mercator's Chart

Mercator's chart represents the surface of the earth as a rectangle. The meridians are represented by straight lines perpendicular to a straight line representing the equator, and the parallels are represented by straight lines parallel to the line representing the equator. On this chart any two meridians intercept equal distances on all the parallels; but on the earth the intercept on any parallel by two meridians is less than the intercept on the equator.

Hence, at no place except the equator does this chart correctly represent the intercepts on the parallels, and the error increases as we approach the poles.

In order to compensate for this error, the distance of each parallel from the equator is proportionally increased.

These augmented latitudes are called **meridional parts**, and are given for every minute of latitude in Table XI.

On account of the spheroidal form of the earth, the development of the formula for the computation of this table is too difficult for presentation in an elementary work.

The *meridional difference of latitude* is the expanded arc of the meridian between two parallels. It may be found by taking the difference or the sum of the meridional parts corresponding to the two latitudes, according as the latitudes are of the same name or of opposite names.

If the earth is regarded as a sphere,

By [4], $\lambda_d = p \sec L.$

But on Mercator's chart, $p = \lambda_d$; that is, p is too great in the ratio of the secant of the latitude. Hence, the intercepts of the parallels on the meridians must be increased in the

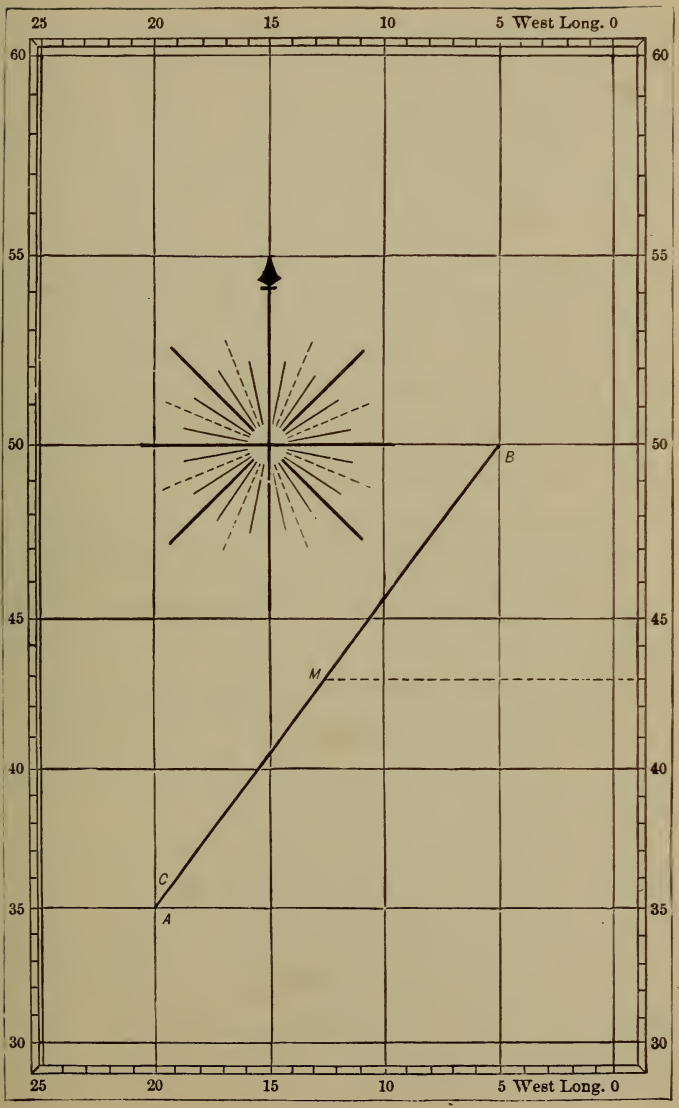


FIG. 21

same ratio. But the intercepts thus augmented must be taken very small, otherwise a great amount of error will be introduced, because the triangles formed by the expanded latitude and the expanded departure cannot be regarded as similar plane triangles unless the distances are small. The augmented latitude for n degrees of latitude would be

$$1'(\sec 1' + \sec 2' + \sec 3' + \cdots + \sec n').$$

The construction and use of Mercator's chart will be understood by referring to Fig. 21, which represents a chart extending from 30° to 60° north latitude, and from 0° to 25° west longitude.

The horizontal line representing the parallel of 30° is first drawn and divided into equal parts. Through the points of division the meridians are drawn perpendicular to the parallel.

In this case each division of the parallel represents $5° = 300'$.

From the Table of Meridional Parts,

Meridional parts corresponding to a latitude of 30° = 1876.9′
Meridional parts corresponding to a latitude of 35° = 2231.1′
Meridional difference of latitude for 5° = 354.2′

Hence, the parallel of 35° is drawn at a distance of 354.2′ above the parallel of 30°.

Meridional parts corresponding to a latitude of 35° = 2231.1′
Meridional parts corresponding to a latitude of 40° = 2607.9′
Meridional difference of latitude for 5° = 376.8′

Hence, the parallel of 40° is drawn at a distance of 376.8′ above the parallel of 35°; and so on.

Let it be required to find from the chart the course and distance by rhumb line from A, in latitude 35° N., longitude 20° W., to B, in latitude 50° N., longitude 5° W.

A is at the intersection of the meridian of 20° and the parallel of 35°; B is at the intersection of the meridian of 5° and the parallel of 50°. Draw AB. AB represents the rhumb line from A to B, for it makes the same angle C with all of the meridians.

The course C may be found by making one edge of a pair of parallel rulers coincide with AB and extending until the other edge passes through the centre of the compass-rose above. The rhumb line of the compass-rose which most nearly coincides with the edge of the ruler shows the course.

The distance may be found by carrying M, the middle point of AB, over to the side, and measuring up and down a distance equal to one half of AB. The number of degrees between the extreme points reduced to miles is the distance required.

It is evident that Mercator's chart shows relative *positions* accurately, but not relative *distances*.

Let CA (Fig. 22) represent the ship's track. CB will represent the true difference of latitude, and BA the departure. Take CE equal to the meridional difference of latitude corresponding to CB, and through E draw ED parallel to BA to meet CA produced in D. Then, CD will represent the ship's track on Mercator's chart, and ED will represent the difference of longitude.

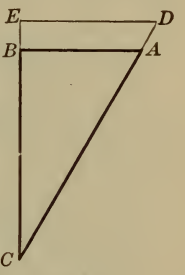


FIG. 22

CE = Mer. L_d = meridional difference of latitude.

$$CB = L_d. \qquad BA = p. \qquad ED = \lambda_d.$$

In the right triangle CED,

$$\tan C = \frac{ED}{CE},$$

or

$$\left.\begin{aligned} \tan C &= \frac{\lambda_d}{\text{Mer. } L_d}; \\ \therefore \lambda_d &= \text{Mer. } L_d \times \tan C. \end{aligned}\right\} \qquad [8]$$

Also, by [3], $\tan C = \dfrac{p}{L_d}$.

$$\therefore \lambda_d = \frac{\text{Mer. } L_d \times p}{L_d}. \qquad [9]$$

Also, by [2], $D = L_d \sec C$; $\therefore L_d = D \cos C$.

To find the difference of longitude made when the ship sails on an oblique course there are two distinct methods: Middle Latitude Sailing and Mercator's Sailing.

When to use Middle Latitude and Mercator's Sailings

Either Middle Latitude or Mercator's Sailing may be used when the distance is small (such as an ordinary day's run); but when the distance is great, Mercator's Sailing should be employed if the course is less than about 45°; and Middle Latitude Sailing should be employed if the course exceeds about 45°.

This distinction is of special importance in high latitudes.

Example 1. A ship sails E.N.E., 200 miles, from latitude 40° 45′ N., longitude 74° W. Find the latitude in and the longitude in.

Solution I. *By computation.*

To find the latitude in:

By [2], $L_d = D \cos C$.

$D = 200$ | $\log D = 2.30103$
$C = 67°\,30'$ | $\log \cos C = 9.58284$
$\log L_d = 1.88387$

$\therefore L_d = 76.5$ miles
$= 1°\,17'$ N.
$L' = 40°\,45'$ N.
$\therefore L'' = 42°\,2'$ N.

To find the longitude in:

$L' = 40°\,45'$ N., Mer. Parts = 2666.8
$L'' = 42°\,2'$ N., Mer. Parts = 2769.0
Mer. L_d = 102.2

By [8], $\lambda_d = \text{Mer. } L_d \times \tan C$.

$\log \text{Mer. } L_d = 2.00945$
$\log \tan C = 10.38278$
$\log \lambda_d = 2.39223$
$\therefore \lambda_d = 246.7$ miles
$= 4°\,7'$ E.
$\lambda' = 74°\,0'$ W.
$\therefore \lambda'' = 69°\,53'$ W.

Solution II. *By inspection.* E.N.E. = 6 points. Corresponding to a course of 6 points and a distance of 200, the true difference of latitude is found in Table VIII to be 76.5; whence, the latitude in and the meridional difference of latitude are found as in Solution I. Corresponding to the course 6 points, and the meridional difference of latitude 102.2 as a difference of latitude, the same table gives the departure 246.7, which is

equal to the difference of longitude required; whence, the longitude in may be found.

EXAMPLE 2. Required the course and the distance from Halifax, in latitude 44° 40′ N. and longitude 63° 35′ W., to Ireland island, in latitude 32° 19′ N. and longitude 64° 49′ W.

$L' = 44° 40'$ N.	Mer. Parts $= 2985.6$	$\lambda' = 63° 35'$ W.
$L'' = 32° 19'$ N.	Mer. Parts $= 2038.6$	$\lambda'' = 64° 49'$ W.
$L_d = 12° 21'$ S.	Mer. $L_d = 947.0$	$\lambda_d = 1° 14'$ W.
$= 741$ miles.		$= 74$ miles.

To find the course:

By [8], $$\tan C = \frac{\lambda_d}{\text{Mer. } L_d}.$$

$\lambda_d = 74$, $\log \lambda_d = 1.86923$

Mer. $L_d = 947$. colog Mer. $L_d = 7.02365 - 10$

$\log \tan C = 8.89288$

$\therefore C =$ S. 4° 28′ W.

To find the distance:

By [2], $$D = L_d \sec C.$$

$L_d = 741$, $\log L_d = 2.86982$

$C = 4° 28'$. $\log \sec C = 0.00132$

$\log D = 2.87114$

$\therefore D = 743.3$ miles.

EXERCISE V

	L'	L''	λ'	λ''	C	D
1	38° 14′ N.	39° 51′ N.	2° 7′ E.	4° 18′ E.	Required	Required
2	49° 53′ N.	48° 28′ N.	6° 19′ W.	5° 3′ W.	Required	Required
3	64° 30′ N.	60° 40′ N.	4° 20′ W.	0° 10′ E.	Required	Required
4	54° 54′ S.	34° 22′ S.	60° 28′ W.	18° 24′ W.	Required	Required
5	17° 0′ N.	20° 0′ N.	180° 0′ E.	177° 0′ E.	Required	Required
6	45° 15′ N.	Required	35° 26′ W.	Required	N. 49° E.	175
7	55° 1′ N.	Required	1° 25′ E.	Required	N. 10° E.	246
8	50° 48′ N.	Required	9° 10′ W.	Required	S. 41° W.	275
9	37° 0′ N.	51° 18′ N.	48° 20′ W.	Required	Required	1027
10	51° 15′ N.	37° 5′ N.	9° 50′ W.	Required	S.W. by S.	Required

11. Required the course and the distance from Toulon to Valencia, by Mercator's Sailing:

Toulon $\begin{cases} L = 43° \ 8' \text{ N.} \\ \lambda = 5° \ 56' \text{ E.} \end{cases}$ Valencia $\begin{cases} L = 39° \ 27' \text{ N.} \\ \lambda = 0° \ 19' \text{ W.} \end{cases}$

12. Required the compass course and the distance from Cape East, New Zealand, to San Francisco; variation 14° 20′ E. and deviation 5° 40′ E.:

Cape East $\begin{cases} L = 37° \ 50' \text{ S.} \\ \lambda = 178° \ 36' \text{ E.} \end{cases}$ San Francisco $\begin{cases} L = 37° \ 48' \text{ N.} \\ \lambda = 122° \ 24' \text{ W.} \end{cases}$

13. Required the course and the distance from Cape Lopatka to Callao:

Cape Lopatka $\begin{cases} L = 51° \ 2' \text{ N.} \\ \lambda = 156° \ 50' \text{ E.} \end{cases}$ Callao $\begin{cases} L = 12° \ 4' \text{ S.} \\ \lambda = 77° \ 14' \text{ W.} \end{cases}$

14. A ship from latitude 20° 40′ N. sails N.E. by N. until she is in latitude 27° 16′ N. Required the distance and the difference of longitude.

15. A ship from Cape Clear, in latitude 51° 26′ N. and longitude 9° 29′ W., sails S.W. by S. until the distance run is 1022 miles. Find the latitude in and the longitude in by Mercator's and Middle Latitude Sailings. Which method is preferable?

SECTION XVI

TRAVERSE SAILING

When a ship reaches her destination by sailing on different courses the difference of latitude may be found by computing the change in latitude for each course separately, as in Plane Sailing, and taking the algebraic sum; or, more conveniently (but less accurately), by writing the north latitudes in one column and the south latitudes in another, finding the sum of each column and taking the arithmetical difference, giving it

the name (N. or S.) of the greater sum. In like manner the total departure may be found approximately.

Evidently the Traverse Table may be used and the work arranged as in Rectangular Surveying.

When the difference of latitude and the departure have been found, the difference of longitude may be determined by Middle Latitude or Mercator's Sailings.

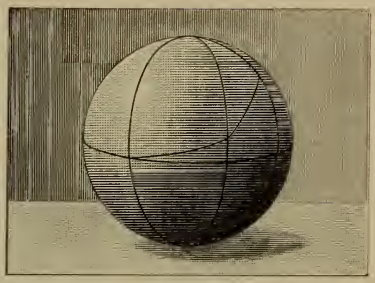


FIG. 23

EXAMPLE 1. A ship in latitude 40° N. and longitude 67° 4′ W. sails N.W. 60 miles, then N. by W. 52 miles, then W.S.W. 83 miles. Find the latitude in and the longitude in.

C		D	N	S	E	W
N.W.	4 points	60	42.4			42.4
N. by W.	1 point	52	51.0			10.1
W.S.W.	6 points	83		31.8		76.7
Hence, L_d = 61.6 miles N.			93.4	31.8	0	129.2
= 1° 2′ N.			31.8			0.
p = 129.2 W.			61.6			129.2

L''

$L' = 40°\ 0'$ N.
$L_d = 1°\ 2'$ N.
$L'' = 41°\ 2'$ N.

L_m

$L' = 40°\ 0'$ N.
$L'' = 41°\ 2'$ N.
$2)81°\ 2'$
$L_m = 40°\ 31'$ N.
co. $L_m = 49°\ 29'$

λ''

$\lambda' = 67°\ 4'$ W.
$\lambda_d = 2°\ 51'$ W.
$\lambda'' = 69°\ 55'$ W.

The difference of longitude may be found by computation, using formula [5], or by inspection, as in Sect. XV, p. 347, Example 2. 49° is found at the bottom of p. 132, Table IX, and opposite the departure 129.1 is distance 171, which is the difference of longitude required. $171' = 2°\ 51'$.

To plot the Courses

Let C (Fig. 24) be the centre of a circle representing the card of the compass. Draw CA, N.W. from C, equal to 60, to any convenient scale. Make an edge of a pair of parallel rulers coincide with the N. by W. rhumb of the compass diagram, extend until the other edge embraces A, and draw AB equal to 52. Make an edge of the ruler coincide with the W.S.W. rhumb of the compass diagram, extend until the other edge embraces B, and draw BD equal to 83. The point D represents the position of the vessel.

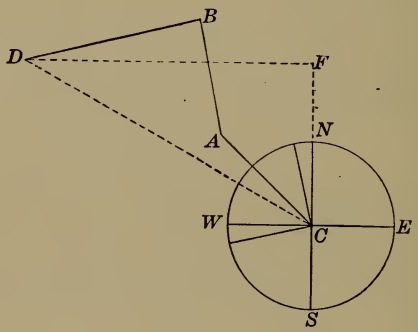


Fig. 24

Upon CN produced let fall the perpendicular DF. Then, CF represents the total difference of latitude, CD the distance made good, and DCF the course made good.

The departure may be regarded as measured on the parallel $\frac{1}{2}(L' + L'')$.

Example 2. The U.S. steamer *Swatara* left latitude 37° 25′ N. and longitude 3° 14′ E. at noon, Feb. 25, 1880, and during the next 24 hours sailed on the (corrected) courses in the

following table. Find the latitude in and the longitude in by account, at noon Feb. 26.

C		D	N	S	E	W
N.E. by E. $\frac{3}{4}$ E.	$5\frac{3}{4}$ points	2.0	0.9		1.8	
E. by N.	7 points	28.0	5.5		27.5	
E. $\frac{1}{2}$ S.	$7\frac{1}{2}$ points	3.0		0.3	3.0	
E.S.E. $\frac{1}{2}$ E.	$6\frac{1}{2}$ points	2.8		0.8	2.7	
S.S.E. $\frac{1}{2}$ E.	$2\frac{1}{2}$ points	3.4		3.0	1.6	
N. by W. $\frac{1}{4}$ W.	$1\frac{1}{4}$ points	4.2	4.1			1.0
N. $\frac{1}{4}$ E.	$\frac{1}{4}$ point	4.4	4.4		0.2	
N. by E. $\frac{1}{4}$ E.	$1\frac{1}{4}$ points	10.6	10.3		2.6	
N. $\frac{1}{2}$ E.	$\frac{1}{2}$ point	11.4	11.4		1.1	
N. $\frac{1}{2}$ W.	$\frac{1}{2}$ point	18.6	18.5			1.8
N. by W.	1 point	34.8	34.1			6.8
S. by E. $\frac{1}{2}$ E.	$1\frac{1}{2}$ points	5.0		4.8	1.5	
			89.2	8.9	42.0	9.6
Hence, $L_d = 80.3 = 1°\,20'$ N.,			8.9		9.6	
and $p = 32.4$ E.			80.3		32.4	

L''

$L' = 37°\,25'$ N.
$L_d = 1°\,20'$ N.
$L'' = 38°\,45'$ N.

L_m

$L' = 37°\,25'$ N.
$L'' = 38°\,45'$ N.
$2)76°\,10'$
$L_m = 38°\,5'$ N.
co. $L_m = 51°\,55'$

λ''

$\lambda' = 3°\,14'$ E.
$\lambda_d = 0°\,41'$ E.
$\lambda'' = 3°\,55'$ E.

EXERCISE VI

Required the difference of latitude and the departure:

1

C	D
S.S.W.	48
S.W. by S.	36
N.E.	24

2

C	D
S. $\frac{1}{2}$ E.	18
S. W. $\frac{1}{2}$ S.	37
S.S.W. $\frac{1}{4}$ W.	56

3

C	D
S.S.W. $\frac{1}{4}$ W.	43
S.S.W. $\frac{1}{2}$ W.	39
S. by W. $\frac{1}{2}$ W.	27

4

C	*D*
N. 25° W.	16.4
N. 8° E.	7.8
N. 19° E.	13.7
N. 76° E.	39.6

5

C	*D*
W.N.W. ¼ W.	21
N.N.E. ¾ E.	9
N. by E. ¾ E.	9
S.S.W. ¼ W.	30

6

C	*D*
S. 83° W.	23.0
S. 48° E.	25.2
N. 48° W.	27.1
N. 36° W.	21.0

7

C	*D*
S. 17° E.	48
S. 45° W.	19
N. 36° W.	18
N. 41° W.	50
E.	36

8

C	*D*
N.N.E.	31
E.N.E.	35
E. by S.	36
S.S.E.	51
S. by E.	60

9

C	*D*
S. 44° E.	69
S. 85° E.	68
S. 27° E.	25
N. 37° W.	5
N. 20° W.	13

SECTION XVII

THE DAY'S WORK

When a ship is starting out upon a voyage, just before she passes out of sight of land the proper officer notes the bearing, and estimates the distance from the ship, of some object on shore whose latitude and longitude are known. This operation is called **taking the departure.** After this has been done, a record is kept on the *log board* of the courses and the distances run. From the log board the record is transferred to the *log book*, in which it is arranged as in the tables on pages 354 and 357.

The column headed H contains the hours from noon to noon. The column headed K contains the knots per hour the ship has sailed; and the next column contains the tenths of knots. The column headed Winds shows the direction of the

wind; the column headed Leeway gives the leeway in points; the column headed Deviation gives the deviation in degrees; and the last column contains a record of miscellaneous events.

From the record of the log book the position of the ship is computed by Traverse Sailing each day at noon, or at any other hour when the position is desired, in the manner described in the examples on pages 356 and 358.

In computing the day's work, the first course of the traverse is found by reversing the departure course. For if, as in Example 1, the bearing of the object *from the ship* is N.E. $\frac{1}{4}$ E. 15 miles, then it follows that the bearing of the ship *from the object on shore* is S.W. $\frac{1}{4}$ W. 15 miles.

This method of finding the place of a ship is subject to many errors. It is impossible to determine the course accurately; the deviation is almost constantly changing; the effect of tides and currents can be estimated only approximately; the log gives the distance roughly; the speed varies during the hour; and so on.

Hence, whenever it is possible to do so the results obtained by dead reckoning should be verified by celestial observation. (See Chapter III.) The results obtained by the latter method should be given the preference.

A number of the following exercises are taken from recent examination papers of the British Marine Board; others are actual transcripts from the log books of vessels.

The log book used in the U.S. navy contains, in addition to the usual data, columns in which are recorded the force of the wind, the height of the barometer, the temperature of the air by dry and wet bulb thermometers, the surface temperature of the water, the state of the weather, the forms of clouds, the proportion of clear sky, the state of the sea, a record of the sail the vessel is under at the end of the watch; also, a record of the miscellaneous events of the day.

EXAMPLE 1.

H	COURSES	K	$\frac{1}{10}$	WINDS	LEEWAY	DEVIATION	REMARKS, ETC.
1	W.S.W.	10	8	N.W.	$\frac{1}{2}$ pt.	11° W.	The departure was taken as follows: A point of land in latitude 37° 3′ N., longitude 9° 0′ W., bearing by compass N E. $\frac{1}{4}$ E., dist. 15 miles. Ship's head W.S.W. Deviation as per log
2		11	4				
3		11	4				
4		11	4				
5	N.W. $\frac{1}{2}$ N.	12	2	W.S.W.	$\frac{1}{4}$ pt.	17° W.	
6		12	3				
7		12	3				
8		12	2				
9	N.N.W.	9	6	W.	$\frac{3}{4}$ pt.	11° W.	
10		9	5				
11		9	5				
12		9	4				Variation 22° 30′ W.
1	N.W. by W.	7	8	S.W. by W.	1$\frac{1}{4}$ pts.	20° W.	
2		7	6				
3		7	4				
4		8	2				A current set the ship S.W. by W. $\frac{3}{4}$ W. (correct magnetic) 8 miles from the time the departure was taken to the end of the day
5	S.W. $\frac{1}{2}$ S.	9	3	S.S.E.	1 pt.	6° W.	
6		8	7				
7		9	3				
8		8	7				
9	W. $\frac{1}{2}$ S.	10	3	S. by W.	$\frac{1}{2}$ pt.	13° W.	
10		10	2				
11		10	2				
12		10	3				

Departure course.

The opposite point to N.E. $\frac{1}{4}$ E. is S.W. $\frac{1}{4}$ W. The ship's head being W.S.W., the deviation is the same as for the first course, 11° W.

S.W. $\frac{1}{4}$ W. = 4$\frac{1}{4}$ pts. R. of S.
or 47° 49′ R. of S.
Dev. 11° 0′ W.
Var. 22° 30′ W. } = 33° 30′ L.
True course = 14° 19′ R. of S.

Hence, the first course and the distance of the traverse are

S. 14° W. 15 miles.

First course.

W.S.W. = 6 pts. R. of S.
Leeway = $\frac{1}{2}$ pt. L.
5$\frac{1}{2}$ pts. R. of S.
or 61° 53′ R. of S.
Dev. 11° 0′ W.
Var. 22° 30′ W. } = 33° 30′ L.
True course = 28° 23′ R. of S.

The distance (45 miles) for this course is found by adding the distances sailed each hour up to 5 o'clock, when the course was changed.

Hence, the second course and the distance of the traverse are

S. 28° W. 45 miles.

Second course.

N.W. ½ N.	= 3½ pts. L. of N.
Leeway	= ¼ pt. R.
	3¼ pts. L. of N.
or	36° 34′ L. of N.
Dev. 17° 0′ W. / Var. 22° 30′ W.	= 39° 30′ L.
True course	= 76° 4′ L. of N.

The distance (49 miles) is obtained by adding the hourly distances from 5 o'clock to 9 o'clock.

Hence, the third course and the distance of the traverse are

N. 76° W. 49 miles.

Third course.

N.N.W.	= 2 pts. L. of N.
Leeway	= ¾ pt. R.
	1¼ pts. L. of N.
or	14° 4′ L. of N.
Dev. 11° 0′ W. / Var. 22° 30′ W.	= 33° 30′ L.
True course	= 47° 34′ L. of N.

The distance (38 miles) is obtained by adding the hourly distances from 9 o'clock to 1 o'clock.

Hence, the fourth course and the distance of the traverse are

N. 48° W. 38 miles.

Fourth course.

N.W. by W.	= 5 pts. L. of N.
Leeway	= 1¼ pts. R.
	3¾ pts. L. of N.
or	42° 11′ L. of N.
Dev. 20° 0′ W. / Var. 22° 30′ W.	= 42° 30′ L.
True course	= 84° 41′ L. of N.

Hence, the fifth course and the distance of the traverse are

N. 85° W. 31 miles.

Fifth course.

S.W. ½ S.	= 3½ pts. R. of S.
Leeway	= 1 pt. R.
	4½ pts. R. of S.
or	50° 38′ R. of S.
Dev. 6° 0′ W. / Var. 22° 30′ W.	= 28° 30′ L.
True course	= 22° 8′ R. of S.

Hence, the sixth course and the distance of the traverse are

S. 22° W. 36 miles.

Sixth course.

W. ½ S.	= 7½ pts. R. of S.
Leeway	= ½ pt. R.
	8 pts. R. of S.
or	90° 0′ R. of S.
Dev. 13° 0′ W. / Var. 22° 30′ W.	= 35° 30′ L.
True course	= 54° 30′ R. of S.

Hence, the seventh course and the distance of the traverse are

S. 55° W. 41 miles.

Current course.

S.W. by W. ⅜ W.	= 5⅜ pts. R. of S.
or	64° 41′ R. of S.
Variation	= 22° 30′ L.
True course	= 42° 11′ R. of S.

Hence, the eighth course and the distance of the traverse are

S. 42° W. 8 miles.

The Traverse

	Courses	Dist.	N	S	E	W
1	S. 14° W.	15		14.6		3.6
2	S. 28° W.	45		39.7		21.1
3	N. 76° W.	49	11.9			47.5
4	N. 48° W.	38	25.4			28.2
5	N. 85° W.	31	2.7			30.9
6	S. 22° W.	36		33.4		13.5
7	S. 55° W.	41		23.5		33.6
8	S. 42° W.	8		5.9		5.4
	$L_d = 77.1$ miles		40.0	117.1		183.8
	$= 1° 17'$ S.			40.0		
	$p = 183.8$ W.			77.1		

To find the latitude in:

$$L' = 37° \ 3' \text{ N.}$$
$$L_d = 1° \ 17' \text{ S.}$$
$$L'' = 35° \ 46' \text{ N.}$$

Also, $L_m = 36° \ 25'$ N.

To find the direct course:

By [3], $\tan C = \dfrac{p}{L_d}$.

$$\log p = 2.26435$$
$$\text{colog } L_d = \underline{8.11295 - 10}$$
$$\log \tan C = 10.37730$$
$$\therefore C = \text{S. } 67° \ 15' \text{ W.}$$

To find the direct distance:

By [1], $D = \dfrac{p}{\sin C}$.

$$\log p = 2.26435$$
$$\text{colog sin } C = \underline{0.03517}$$
$$\log D = 2.29952$$
$$\therefore D = 199 \text{ miles.}$$

To find the difference of longitude:

By [5], $\lambda_d = p \sec L_m$.

$$\log p = 2.26435$$
$$\log \sec L_m = \underline{0.09435}$$
$$\log \lambda_d = 2.35870$$
$$\therefore \lambda_d = 228' = 3° \ 48' \text{ W.}$$

To find the longitude in:

$$\lambda' = 9° \ 0' \text{ W.}$$
$$\lambda_d = 3° \ 48' \text{ W.}$$
$$\lambda'' = 12° \ 48' \text{ W.}$$

EXAMPLE 2.

H	COURSES	K	$\frac{1}{16}$	WINDS	LEEWAY	DEVIATION	REMARKS
1	S.W.	6	0	W.N.W.	1 pt.	0	Latitude from 28° 30′ S. Longitude from 10° 15′ E.
2		6	3				
3		6	4				
4		6	0				
5		5	3				
6	S. by W.	6	0	W. by S.	1 pt.		
7		5	1				
8		5	4				
9		5	2				
10		5	3				Variation 1½ pts. W.
11		5	0				
12	S.S.W.	5	2	W.	1 pt.		
1		5	2				
2		5	0				
3		4	6				A current set S. (correct magnetic) 2½ miles per hour from noon to noon
4		5	0				
5		5	0				
6	S.E. by S.	5	2	S.W. by S.	1 pt.		
7		5	4				
8		5	5				
9		6	0				
10		6	0				
11		5	4				
12		5	5				

First course.

S.W. = 4 pts. R. of S.
Leeway = 1 pt. L.
3 pts. R. of S.
Variation = 1½ pts. L.
True course = 1½ pts. R. of S.
= S. 1½ pts. W.

Second course.

S. by W. = 1 pt. R. of S.
Leeway = 1 pt. L.
0 pts. R. of S.
Variation = 1½ pts. L.
True course = 1½ pts. L. of S.
= S. 1½ pts. E.

Third course.

S.S.W.	= 2 pts. R. of S.
Leeway	= 1 pt. L.
	1 pt. R. of S.
Variation	= 1½ pts. L.
True course	= ½ pt. L. of S.
	= S. ½ pt. E.

Fourth course.

S.E. by S.	= 3 pts. L. of S.
Leeway	= 1 pt. L.
	4 pts. L. of S.
Variation	= 1½ pts. L.
True course	= 5½ pts. L. of S.
	= S. 5½ pts. E.

Current course.

S.	= 0 pts. L. of S.
Variation	= 1½ pts. L.
True course	= 1½ pts. L. of S.
	= S. 1½ pts. E.

THE TRAVERSE

	Courses	Dist.	N	S	E	W
1	S. 1½ points W.	30		28.7		8.7
2	S. 1½ points E.	32		30.6	9.3	
3	S. ½ point E.	30		29.9	2.9	
4	S. 5½ points E.	39		18.4	34.4	
5	S. 1½ points E.	60		57.4	17.4	
L_d = 165 miles = 2° 45′ S.				165.0	64.0	8.7
L' = 28° 30′ S.					8.7	
L'' = 31° 15′ S.					55.3	

L' = 28° 30′ S.
L'' = 31° 15′ S.

Mer. Parts = 1774.3
Mer. Parts = 1963.6
Mer. L_d = 189.3

To find the difference of longitude:

By [9], $$\lambda_d = \frac{\text{Mer. } L^d \times p}{L_d}.$$

Mer. L_d = 189.3, p = 55.3, L_d = 165.0.

log Mer. L_d = 2.27715
log p = 1.74273
colog L_d = 7.78252 − 10
log λ_d = 1.80240
$\therefore \lambda_d$ = 63.4 = 1° 3′ E.

λ' = 10° 15′ E.
λ_d = 1° 3′ E.
λ'' = 11° 18′ E.

NOTE. Find the course by [3] to determine whether to use Middle Latitude or Mercator's Sailing.

The distance made good may be found by [2].

EXERCISE VII

Find the latitude in and the longitude in from the following log-book records:

1

H	Courses	K	T	Winds	Leeway	Deviation	Remarks
1	N.N.E.	6	3	W.	0	0	Latitude from 46° 28′ N. Longitude from 22° 18′ W.
2		6	2				
3		6	5				
4		6	4				
5		6	0				
6	E.N.E.	6	1	N.W.			
7		6	6				
8		5	8				
9		5	6				
10		5	4				
11		5	5				
12	E. by S.	5	3	N.			Variation 0
1		5	9				
2		6	2				
3		6	0				
4		6	3				
5		6	4				
6	S.S.E.	7	0	N.			The tide set S. by E. 2¼ miles per hour during the 24 hours
7		6	8				
8		7	3				
9		7	5				
10		7	1				
11		7	9				
12		7	3				

2

H	Courses	K	T	Winds	Leeway	Deviation	Remarks
1	S. by W.	6	2	W.	0	0	Latitude from 33° 40′ N. Longitude from 16° 20′ W.
2		6	0				
3		6	3				
4		7	0				
5		7	2				
6		7	3				
7	S.W. by S.	7	2	W. by N.	1 pt.		
8		7	2				
9		7	4				
10		7	6				
11		7	4				
12		8	1				Variation 12° 20′ W.
1		8	0				
2		8	5				
3		8	2				
4	S.W. by W.	7	5	N.W.	0		
5		7	3				
6		6	6				A current set W.S.W. (correct magnetic) 1½ miles per hour from noon to noon
7		6	4				
8		6	0				
9		6	2				
10		6	1				
11		6	3				
12		6	1				

3

H	Courses	K	T	Winds	Leeway	Deviation	Remarks
1	N. by E.	6	7	E. by N.	1 pt.	0	Latitude from 1° 30′ S. Longitude from 0° 10′ E.
2		6	2				
3		6	4				
4		6	3				
5		6	1				
6		6	0				
7	N.	5	8	E.N.E.	1 pt.		
8		5	4				
9		5	0				
10		5	3				
11		5	6				
12		5	9				Variation 13° 30′ W.
1		5	7				
2	N.N.W.	6	4	N.E.	1 pt.		
3		6	8				
4		7	0				
5		7	3				
6		7	6				The current set W. N.W. (correct magnetic) ½ mile per hour from noon to noon
7		7	5				
8		7	0				
9		7	2				
10		7	4				
11		6	3				
12		6	0				

4

H	Courses	K	T	Winds	Leeway	Deviation	Remarks
1	S.E. by E.	13	2	N.	0	11° E.	A point of land, latitude 47° 31′ N., longitude 52° 33′ W., bearing by compass W.S.W., distant 18 miles. Ship's head S.E. by E. Deviation as per log
2		13	3				
3		12	5				
4		13					
5	S.E.	11		E.N.E.	½ pt.	9° E.	
6		10	5				
7		10	4				
8		11	1				
9	E. by N.	8	2	S.E. by S.	1 pt.	17° E.	
10		9	4				
11		9	4				
12		9					
1	E.N.E.	6	8	S.E.	1½ pts.	15° E.	Variation 28° W.
2		6	7				
3		6	5				
4		7					
5	S.S.E.	5	8	E.	2 pts.	7° E.	A current set the ship (correct magnetic) S. by E. 12 miles from the time the departure was taken to the end of the day
6		5	8				
7		6	4				
8		6					
9	S.E. by S.	7		E. by N.	1¼ pts.	8° E.	
10		7	3				
11		7	4				
12		7	3				

5

H	Courses	K	T	Winds	Leeway	Deviation	Remarks
1	S.S.E.	5	6	E.	2¼ pts.	3° E.	A point of land in latitude 62° N., longitude 150° E., bearing by compass W. by S. ¼ S., distant 17 miles. Ship's head S.S.E. Deviation as per log
2		5	6				
3		5	0				
4		4	8				
5	S.S.W. ½ W.	4	7	W.	2¾ pts.	4° W.	
6		4	8				
7		5	2				
8		5	3				
9	W.S.W.	5	0	S.	2½ pts.	9° W.	
10		6	0				
11		6	5				
12		6	5				
1	W. ½ N.	6	6	N. by E.	0	11° W.	Variation 31° E.
2		7	0				
3		6	4				
4		6	0				
5	E.	4	6	S.S.E.	2½ pts.	10° E.	
6		5	0				A current set the ship (correct magnetic) N.N.E. 21 miles from the time the departure was taken to the end of the day
7		4	8				
8		4	6				
9	E.S.E.	4	0	S. by W.	0	9° E.	
10		4	5				
11		4	5				
12		5	0				

6

H	Courses	K	T	Winds	Leeway	Deviation	Remarks
1	S.W. ½ W.	14	5	S.E.	0	6° W.	A point, latitude 50° 12′ S., longitude 179° 40′ W., bearing by compass N. ¾ W., distant 19 miles. Ship's head S.W. ½ W. Deviation as per log
2		14	2				
3		14	6				
4		14	7				
5	N. ⅜ E.	4	0	E.N.E.	3¼ pts.	3° E.	
6		3	6				
7		3	6				
8		3	8				
9	S. by E. ½ E.	2	4	S.W. ½ W.	2⅜ pts.	6° E.	
10		2	3				
11		2	3				
12		2	0				
1	W. by S.	12	2	S. by W.	¼ pt.	14° W.	Variation 14° E.
2		12	4				
3		12	6				
4		12	8				
5	E.N.E.	3	0	S.E.	2½ pts.	19° E.	A current set the ship (correct magnetic) S.W. ¼ W. 42 miles from the time the departure was taken to the end of the day
6		2	3				
7		3	4				
8		3	3				
9	S.S.W. ½ W.	5	6	S.E.	1⅜ pts.	4° W.	
10		5	7				
11		5	3				
12		5	4				

7

H	Courses	K	T	Winds	Leeway	Deviation	Remarks
1							Left Malaga at 8 P.M. { L 36° 42′ N. λ 4° 25′ W.
2							
3							
4							
5							
6							
7							
8	S.E.	1	5		0	¼ pt. E.	
9		6	4				
10		6	6				
11		6	8				
12		6	5				
1	E.S.E. ¼ E.	7	2		0	0	Variation 1¾ pts. W.
2		7	2				
3		8	2				
4		8	6				
5		8	0				
6		7	6				
7		7	6				
8		7	6				
9		7	5				
10		5	7				
11	E.	4	2		0	½ pt. E.	
12		4	5				

SECTION XVIII

GREAT CIRCLE SAILING

It may be proven (*Geometry*, § 816) that the shortest distance between two places on the surface of the earth is the less of the two arcs of a great circle lying between these places. The rhumb line is the arc of a great circle only when it coincides with the equator or a meridian.

The advantages of great circle sailing are:

(1) The distance is less than by any other route.

(2) The great circle track gives the real direction of the desired port, and this frequently determines which tack the ship should be put on.

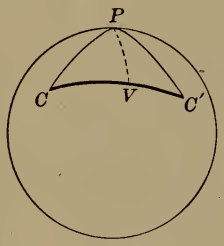


FIG. 25

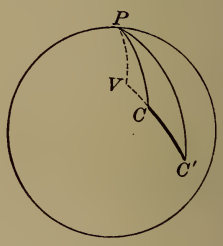


FIG. 26

To sail on a great circle would require a constant change of course. This difficulty is practically overcome by sailing on rhumb lines from point to point of the great circle.

The course of the vessel may be said to describe part of the perimeter of a polygon with a large number of sides within the arc of a great circle.

Definition. The *vertex* of a great circle track is the point nearest the pole and therefore farthest from the equator, and touches the highest latitude. Hence, the vertex is the foot of the perpendicular let fall upon the great circle from the

pole, because the perpendicular measures the shortest distance. When the two places are widely separated, the vertex generally lies between the two places, but it by no means necessarily does lie between the two places. It will be found that if the two angles C and C' (Figs. 25 and 26) are both greater than 90° or both less than 90°, the vertex will fall between the two places; otherwise it will not.

The best practical method of finding the courses, distance, etc., of the great circle track is by the use of Godfray's Course and Distance Diagram, which affords a simple graphic solution.

The solution by Spherical Trigonometry is given here.

FIG. 27

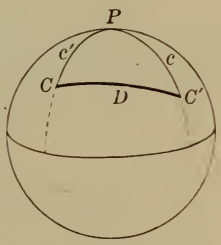


FIG. 28

Let CC' (Fig. 28) represent the great circle track between C and C', P the pole of the equator, and PC and PC' meridians.

C = the first course from C.

C' = the first course from C'.

$P = \lambda_d$ = the difference of longitude of C and C'.

$PC = c'$ = the polar distance of $C = 90° \mp$ latitude of C.

$PC' = c$ = the polar distance of $C' = 90° \mp$ latitude of C'.

$CC' = D$ = the length of the great circle track.

In Navigation two sides of the spherical triangle are the two meridians passing through the places between which the ship's

track lies, and the third side is the ship's track. The lengths of the two sides formed by the meridians are always known from the latitudes of the places to and from; one of the angles, that at the pole, is found from the difference of longitude between the two places. Hence, it is evident we have to solve a spherical triangle, given *two sides and the included angle.*

The solution consists in determining the following elements:

(1) The first and last courses.
(2) The distance.
(3) The position (latitude and longitude) of the vertex.
(4) A succession of points on the great circle.

To find the First Course

In the triangle CPC' (Fig. 30) by Napier's Analogies (*Spherical Trigonometry*, Sect. LVI, p. 164):

$$\left.\begin{aligned} \tan \tfrac{1}{2}(C + C') &= \frac{\cos \tfrac{1}{2}(c - c')}{\cos \tfrac{1}{2}(c + c')} \cot \tfrac{1}{2}\lambda_d, \\ \tan \tfrac{1}{2}(C - C') &= \frac{\sin \tfrac{1}{2}(c - c')}{\sin \tfrac{1}{2}(c + c')} \cot \tfrac{1}{2}\lambda_d, \\ C &= \tfrac{1}{2}(C + C') + \tfrac{1}{2}(C - C'), \\ C' &= \tfrac{1}{2}(C + C') - \tfrac{1}{2}(C - C'). \end{aligned}\right\} \quad [10]$$

To find the Distance

By Gauss's Equations (*Spherical Trigonometry*, Sect. LVI, p. 164):

$$\cos \tfrac{1}{2} D = \frac{\cos \tfrac{1}{2}(c + c')}{\cos \tfrac{1}{2}(C + C')} \sin \tfrac{1}{2}\lambda_d. \quad [11]$$

(See also *Spherical Trigonometry*, Sect. LXV, p. 182.)

The position of the vertex must be known to enable us to calculate the course and the distance from one point to another on the great circle.

To find the Latitude of the Vertex

Let PV (Fig. 30) be the perpendicular from P upon the arc of a great circle CC'. Then V is the vertex of CC'.

In the right triangle CVP,

$$\sin PV = \cos(co.\ c')\cos(co.\ C).$$

Hence,
$$\left.\begin{aligned}\textbf{sin PV} &= \textbf{sin c' sin C},\\ \textbf{Latitude of V} &= \textbf{90° — PV.}\end{aligned}\right\} \quad [12]$$

(See *Spherical Trigonometry*, Sect. LI, p. 147.)

To find the Longitude of the Vertex from C

We use the known elements c' and C to find CPV.

In the right triangle CVP (Fig. 30),

$$\sin(co.\ c') = \tan(co.\ C)\tan(co.\ CPV).$$
$$\therefore \cos c' = \cot C \cot CPV.$$

Hence, $$\textbf{cot CPV} = \textbf{cos c' tan C}. \quad [13]$$

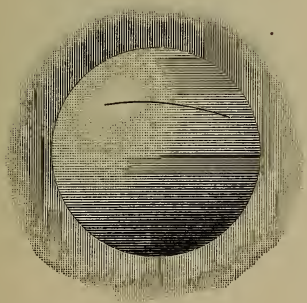


Fig. 29

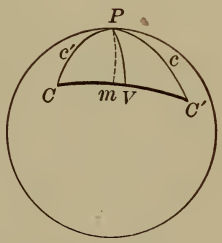


Fig. 30

By combining this result with the longitude of C, the longitude of V may be found.

It is evident that if a series of points is laid down on the great circle track and the navigator sails the ship from point

to point by Mercator's or Middle Latitude Sailing, he will shorten his route very considerably. It is easy to find a succession of points, beginning at the vertex, since the angle at the vertex is a right angle. Hence, knowing the latitude and the longitude of different points on the track the navigator may sail from one point to another.

To find the Latitude of Intermediate Points of Given Longitude, in order to plot the Great Circle Track on the Chart

Let m (Fig. 31) be a point whose longitude from V is assumed.

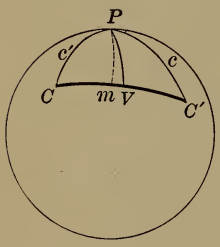


Fig. 31

In the right triangle PVm,

$$\sin(co.\ mPV) = \tan PV \tan(co.\ Pm).$$
$$\therefore \cos mPV = \tan PV \cot Pm.$$

Hence,

$$\left.\begin{aligned} \textbf{cot Pm} &= \textbf{cos mPV cot PV},\\ \textbf{Latitude of m} &= \pm\,(90^\circ - \textbf{Pm}),\end{aligned}\right\} \quad [14]$$

$\pm$ according as m is above or below the equator.

Points m, etc., are usually assumed on each side of V, differing from V and from each other 5° or 10° in longitude. In this manner the latitude and longitude of as many points as we please between C and C' may be determined. The course and distance from point to point by rhumb line may then be found by Middle Latitude or Mercator's Sailings.

When either of the angles C or C' is greater than 90° and the other is less than 90°, the vertex will be on CC' produced through the vertex of the angle which exceeds 90°. (See Fig. 26, p. 366.)

Note. If L' and L'' represent the latitudes of C and C', respectively, the first formula of set [10] becomes

$$\tan \tfrac{1}{2}(C + C') = \frac{\cos \tfrac{1}{2}(L' - L'')}{\sin \tfrac{1}{2}(L' + L'')} \cot \tfrac{1}{2} \lambda_d,$$

provided both places are on the same side of the equator. The student will find it a valuable exercise to express all of the preceding formulas of this section similarly. When both C and C' are south of the equator the polar distance must be reckoned from the south pole.

To plot the Great Circle Track on Mercator's Chart

(1) By means of great circle charts. On these charts the great circle is represented by a straight line; hence, the latitude of each intersection with the meridians is known. The corresponding points may be marked on the meridians on Mercator's chart and a line traced through these points. This curved line will represent the great circle track.

(2) By computing the latitudes corresponding to certain longitudes (or *vice versa*) of the great circle, marking these points on Mercator's chart, and tracing the curve through them.

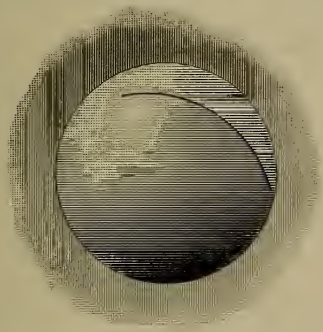


Fig. 32

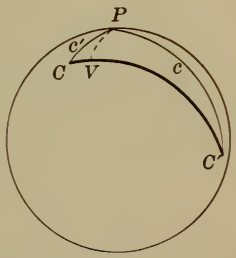


Fig. 33

Example 1. Required the initial courses, the distance, and the latitude and longitude of the vertex of the great circle track joining C and C'. Given:

Lat. of C = 54° 18′ N.	Long. of C = 142° 38′ E.
Lat. of C' = 7° 12′ S.	Long. of C' = 90° 0′ W.

Referring to the triangle CPC' (Fig. 33),

$$c = 90° + 7° 12' = 97° 12',$$
$$c' = 90° - 54° 18' = 35° 42',$$
$$\lambda_d = P = (180° - 142° 38') + 90° = 127° 22'.$$

To find the initial courses:

By [10], $$\tan \tfrac{1}{2}(C + C') = \frac{\cos \tfrac{1}{2}(c - c')}{\cos \tfrac{1}{2}(c + c')} \cot \tfrac{1}{2}\lambda_d,$$

and $$\tan \tfrac{1}{2}(C - C') = \frac{\sin \tfrac{1}{2}(c - c')}{\sin \tfrac{1}{2}(c + c')} \cot \tfrac{1}{2}\lambda_d.$$

$\frac{1}{2}(c-c') = 30° 45'$	$\log \cos \frac{1}{2}(c - c') = 9.93420$	$\log \sin \frac{1}{2}(c-c') = 9.70867$
$\frac{1}{2}(c+c') = 66° 27'$	$\operatorname{colog} \cos \frac{1}{2}(c+c') = 0.39843$	$\operatorname{colog} \sin \frac{1}{2}(c+c') = 0.03777$
$\frac{1}{2}\lambda_d = 63° 41'$	$\log \cot \frac{1}{2}\lambda_d = 9.69425$	$\log \cot \frac{1}{2}\lambda_d = 9.69425$
	$\log \tan \frac{1}{2}(C + C') = 10.02688$	$\log \tan \frac{1}{2}(C - C') = 9.44069$
	$\frac{1}{2}(C + C') = 46° 46'$	$\frac{1}{2}(C - C') = 15° 25'$

$\therefore C = 62° 11'$, or N. 62° 11' E.;

and $C' = 31° 21'$, or N. 31° 21' W.

To find the distance:

By [11], $$\cos \tfrac{1}{2} D = \frac{\cos \tfrac{1}{2}(c + c')}{\cos \tfrac{1}{2}(C + C')} \sin \tfrac{1}{2}\lambda_d.$$

$\frac{1}{2}(c + c') = 66° 27'$	$\log \cos \frac{1}{2}(c + c') = 9.60157$
$\frac{1}{2}(C + C') = 46° 46'$	$\operatorname{colog} \cos \frac{1}{2}(C + C') = 0.16433$
$\frac{1}{2}\lambda_d = 63° 41'$	$\log \sin \frac{1}{2}\lambda_d = 9.95248$
	$\log \cos \frac{1}{2} D = 9.71838$
	$\frac{1}{2} D = 58° 29'$
	$\therefore D = 116° 58'$
	$= 7018$ miles.

To find the latitude of the vertex:

By [12], $\sin PV = \sin c' \sin C$.

$c' = 35° 42'$	$\log \sin c' = 9.76607$
$C = 62° 11'$	$\log \sin C = 9.94667$
	$\log \sin PV = 9.71274$
	$PV = 31° 4'$
L of $V = 90° - 31° 4' = 58° 56'$ N.	

To find the longitude of the vertex:

By [13],

$$\cot CPV = \cos c' \tan C.$$

	$\log \cos c' = 9.90960$
	$\log \tan C = 10.27769$
	$\log \cot CPV = 10.18729$
	$CPV = 33° 1'$
	λ of $C = 142° 38'$ E.
	λ of $V = 175° 39'$ E.

Example 2. The Cunard steamers in sailing from New York to Liverpool keep approximately on the great circle between a point off the American coast in latitude 42° N., longitude 50° W., and Fastnet Rock off the Irish coast, in latitude 51° 24′ N., longitude 9° 36′ W. Required the initial course, the distance between these points, and the latitude and longitude of the vertex.

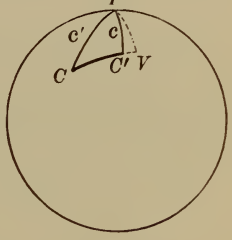


Fig. 34

In the triangle CPC' (Fig. 34) let C represent the initial point, C' Fastnet, and P the pole.

$c' = 90° - 42° 0'$
$= 48° 0' =$ colatitude of C,
$c = 90° - 51° 24'$
$= 38° 36' =$ colatitude of C',
$\lambda_d = P = 50° - 9° 36' = 40° 24'$
$=$ difference of longitude of C and C'.

To find the initial course:

By [10], $$\tan \tfrac{1}{2}(C' + C) = \frac{\cos \tfrac{1}{2}(c' - c)}{\cos \tfrac{1}{2}(c' + c)} \cot \tfrac{1}{2}\lambda_d,$$

and $$\tan \tfrac{1}{2}(C' - C) = \frac{\sin \tfrac{1}{2}(c' - c)}{\sin \tfrac{1}{2}(c' + c)} \cot \tfrac{1}{2}\lambda_d.$$

$\frac{1}{2}(c'-c) = 4° 42'$	$\log\cos\frac{1}{2}(c'-c) = 9.99854$	$\log\sin\frac{1}{2}(c'-c) = 8.91349$
$\frac{1}{2}(c'+c) = 43° 18'$	$\operatorname{colog}\cos\frac{1}{2}(c'+c) = 0.13800$	$\operatorname{colog}\sin\frac{1}{2}(c'+c) = 0.16379$
$\frac{1}{2}\lambda_d = 20° 12'$	$\log\cot\frac{1}{2}\lambda_d = 10.43424$	$\log\cot\frac{1}{2}\lambda_d = 10.43424$
	$\log\tan\frac{1}{2}(C'+C) = 10.57078$	$\log\tan\frac{1}{2}(C'-C) = 9.51152$
	$\frac{1}{2}(C'+C) = 74° 58'$	$\frac{1}{2}(C'-C) = 17° 59'$

$\therefore C = 56° 59'$, or N. 56° 59′ E.

To find the distance:

By [11], $$\cos \tfrac{1}{2} D = \frac{\cos \tfrac{1}{2}(c' + c)}{\cos \tfrac{1}{2}(C' + C)} \sin \tfrac{1}{2}\lambda_d.$$

$\frac{1}{2}(c' + c) = 43° 18'$	$\log\cos\frac{1}{2}(c' + c) = 9.86200$
$\frac{1}{2}(C' + C) = 74° 58'$	$\operatorname{colog}\cos\frac{1}{2}(C' + C) = 0.58606$
$\frac{1}{2}\lambda_d = 20° 12'$	$\log\sin\frac{1}{2}\lambda_d = 9.53819$
	$\log\cos\frac{1}{2} D = 9.98625$
	$\frac{1}{2} D = 14° 21'$
	$\therefore D = 28° 42'$
	$= 1722$ miles.

To find the latitude of the vertex:

By [12], $\sin PV = \sin c' \sin C$.

$\log \sin c' = 9.87107$
$\log \sin C = 9.92351$
$\log \sin PV = 9.79458$

$PV = 38° 33'$
L of $V = 90° - 38° 33' = 51° 27'$ N.

To find the longitude of the vertex:

By [13], $\cot CPV = \cos c' \tan C$.

$\log \cos c' = 9.82551$
$\log \tan C = 10.18721$
$\log \cot CPV = 10.01272$

$CPV = 44° 10'$
λ of $C = 50° 0'$ W.
λ of $V = 5° 50'$ W.

NOTE. $\frac{1}{2}(C' + C) = 74° 57' 41''$; using this value, $D = 1730$ miles. From Sandy Hook to the company's position is 1088 miles; thence to Fastnet by great circle, 1730 miles; total, 2818 miles. The routes recommended by Captain Maury are mainly great circle routes.

EXERCISE VIII

1. Find the elements (initial courses, distance, and latitude and longitude of the vertex) of the great circle track between the Lizard, in latitude 49° 58′ N., longitude 5° 12′ W., and the Bermuda Islands, in latitude 32° 18′ N., longitude 64° 50′ W.

2. Find the elements of the great circle track between Boston (Minot's Ledge lighthouse), in latitude 42° 16′ N., longitude 70° 46′ W., and Cape Clear, in latitude 51° 26′ N., longitude 9° 29′ W. (Take $\frac{1}{2}\lambda_d = 30° 39'$.)

3. Find the elements of the great circle track between Vancouver Island, in latitude 50° N., longitude 128° W., and Honolulu, in latitude 21° 18′ N., longitude 157° 52′ W.

4. Find the elements of the great circle track between Cape Clear, in latitude 51° 26′ N., longitude 9° 29′ W., and Sandy Hook, in latitude 40° N., longitude 74° W.

5. Find the elements of the great circle track between the Lizard, in latitude 49° 58′ N., longitude 5° 12′ W., and Cape Frio, in latitude 23° S., longitude 42° W.

6. Find the elements of the great circle track between Cape Frio and Cape of Good Hope, in latitude 34° 20′ S., longitude 18° 30′ E. (Reckon from the nearest pole.)

7. Find the elements of the great circle track between Grand Port, Mauritius, in latitude 20° 24′ S., longitude 57° 47′ E., and Perth, in latitude 32° 3′ S., longitude 115° 45′ E.

8. Find the elements of the great circle track between A, in latitude 16° 38′ N., longitude 70° 55′ W., and B, in latitude 48° 2′ N., longitude 4° 35′ W.

9. A ship sails from A, in latitude 40° S., longitude 148° 30′ E., to B, in latitude 12° 4′ S., longitude 77° 14′ W. Compare the great circle track and the rhumb line between A and B.

CHAPTER III

NAUTICAL ASTRONOMY

SECTION XIX

THE OBSERVED ALTITUDE OF A HEAVENLY BODY

The Sextant

The sextant is an instrument used by navigators for determining the angular distance between two heavenly bodies, or the angular altitude of a heavenly body above the horizon.

It is held in the hand, and by a turn of the wrist the observer can keep the plane of the instrument in the plane of the two bodies whose angular distance he wishes to measure. The angular distance is measured not by pointing first on one and then on the other, but by sighting them both simultaneously and in apparent coincidence. This can be done even when the observer has no fixed position or stable footing.

This instrument is shown in outline in Fig. 35. It consists of a frame ABC in the form of a sector of radius 5 to 8 inches whose arc AB is 60° in extent, but divided into 120 equal parts reckoned as degrees, and each so-called degree is divided into six equal parts. CE is a movable radius which carries at C a mirror called the *index glass*, whose plane is perpendicular to the plane of the sextant; and at E an *index* which moves over the arc AB. At D is fixed the *horizon glass*, also perpendicular to the plane of the sextant and parallel to CA; that is, parallel to the index glass when the index is at 0 (A).

Only one half of the horizon glass is silvered, the upper half being left transparent. F is a telescope fixed to CA, with its axis directed nearly to the line which separates the silvered and unsilvered parts of the horizon glass.

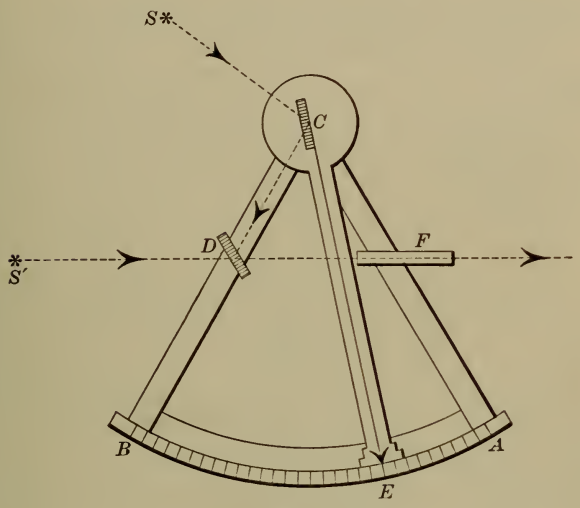


FIG. 35

To determine the angular distance between two heavenly bodies S and S' (Fig. 35), the sextant is held in the hand so that a ray of light from S' passes through the unsilvered part of the horizon glass, enters the telescope, and is seen as if by direct vision. The arm CE is then moved so that a ray of light from S being reflected by the mirrors at C and D, enters the telescope, taking the path $SCDF$. The movement of CE is continued until the image of S appears to be in contact with S', when the arc AE gives the angular distance required.

The angle between the objects whose images coincide is twice the angle between the mirrors.

By the law of reflection, $a = a$, and $b = b$ (Fig. 36).

$$Q = a - b.$$

$$M = 2a - 2b = 2Q = 2Q' = 2ACE.$$

Half degrees on AB are marked as degrees. Hence, AE is the required reading.

If S' is the point on the sea horizon directly under S, the arc AE will give the *observed altitude* of S. The altitude of the sun is measured by pointing the telescope at the sea horizon, which appears like a line in the field of the telescope, and by moving the index arm till one edge of the reflected image of the sun coincides with the horizon. The arc read from the sextant at this time is the sun's altitude. When the altitude of the sun is observed it has to be corrected for *index error*, *refraction*, *dip*, *parallax*, and *semidiameter*.

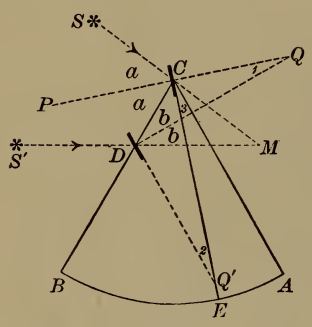


FIG. 36

The **Quintant** has an arc of 72°, and is therefore capable of measuring angles up to 144°. In other respects it is like the sextant.

The **Octant** (commonly, but improperly, called the *quadrant*) has an arc of 45°, and is therefore capable of measuring angles up to 90°. In general construction it is similar to the sextant, for which it is a cheap substitute. It is much more easily handled, and is yet sufficiently accurate for many purposes.

In Fig. 35 the arc AB, extending from zero to the left, is called the *arc proper;* the arc to the right of zero is called the *arc of excess.* The index is *on* the arc or *off* the arc according as it is on the arc proper or on the arc of excess.

When the index glass is parallel to the horizon glass the index should be at 0°. If it is not, the arc between the zero

point and the index is called the **index error.** The index correction is readily discovered as follows:

Bring the direct and reflected images of a star into coincidence and read off the arc. The index correction is numerically equal to this reading, and is positive or negative according as the reading is on the right or left of the zero. The sea horizon may be used instead of a star.

In a perfect instrument the vernier should read strictly 0° when the mirrors are parallel; but however nicely a sextant may be constructed, there is always a small amount of error in the parallelism of the index and horizon glasses.

SECTION XX

CORRECTIONS OF THE OBSERVED ALTITUDE

Refraction

The air which surrounds the earth gradually decreases in density as we ascend from the surface of the earth. At the height of four miles its density is only half as great as at the earth's surface. According to a fundamental law of optics, if a ray of light passes from a rarer to a denser medium, it is bent toward the perpendicular to the surface which separates the two media; but if it passes from a denser to a lighter medium, it is bent from that perpendicular. This law is true when the surface is curved as well as when it is plane.

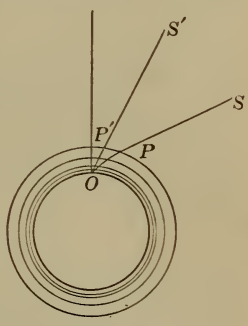


Fig. 37

Let S (Fig. 37) represent the true position of a star, O the position of the observer on the surface of the earth, and PP' the upper limit of the atmosphere.

A ray of light from S meeting the atmosphere at P will be bent toward the perpendicular to the arc PP' at P. As the density of the atmosphere increases from P to O, PO will be a curve, and the star will be seen in the direction OS' tangent to PO at O, the last direction of the ray being that of a tangent to the curved path at the eye of the observer. These laws are here assumed. The facts and reasoning on which they depend belong to works on Optics.

Astronomical refraction is the difference of direction of the two lines PS and OS' (Fig. 37).

The effect of refraction is to increase the apparent altitude of a celestial object, and is greatest when the body is near the horizon, and decreases as the altitude of the body increases.

At the zenith the refraction is nothing. The less the altitude, the more obliquely the rays enter the atmosphere and the greater is the refraction.

Table XIII, constructed partly from observation and partly from theory, gives the correction for mean refraction corresponding to every apparent altitude from 0° to 90°. This correction must evidently be subtracted from the apparent altitude.

The effect, therefore, of refraction is to increase the altitude of a star without affecting its azimuth.

The refraction varies slightly according to the condition of the atmosphere, but this variation is not taken into account in this chapter.

Dip of the Horizon

Let BH'' (Fig. 38) represent the level of the sea, O the position of the observer at the height BO above the sea level, C the centre of the earth, and H'' the farthest point on the surface of the earth visible from O. On account of refraction the visual ray $H''O$ will be a curve concave toward C, and H'' will be seen in the direction of OH' tangent to $H''O$ at O, and

OH' will represent the visible horizon. Let OH, perpendicular to OC, represent the true horizon, from which the apparent altitude of the body is estimated.

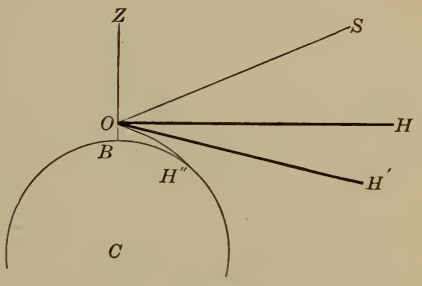


FIG. 38

When we measure the altitude of S, the sun, we measure the angle SOH'; but the altitude we require is the angle SOH: hence, we measure too much by the angle HOH'. The visible horizon has dipped below the rational horizon by the angle HOH', called the *Dip of the Horizon.* We see then that the dip must be subtracted from the observed altitude, and that its amount depends entirely upon the height of the observer.

The dip in minutes of arc is very nearly proportional to the square root of the height of the observer in feet above the sea level.

This element is found in Table XV, which gives the dip corresponding to heights up to 100 feet. The angle HOH' is small even up to 100 feet, being for that height 9′ 48″. Navigators on the decks of their vessels are rarely over 25 feet above the sea level.

PARALLAX

Observations of the celestial bodies comparatively near the earth, when made at different places, do not give the same result, because they are viewed from different points. To correct this difference, astronomers have agreed to reduce the observations to what they would be if they had been made at the centre of the earth.

The difference between the positions of a heavenly body as seen from the centre of the earth and a point on its surface at the same time is called *geocentric parallax.*

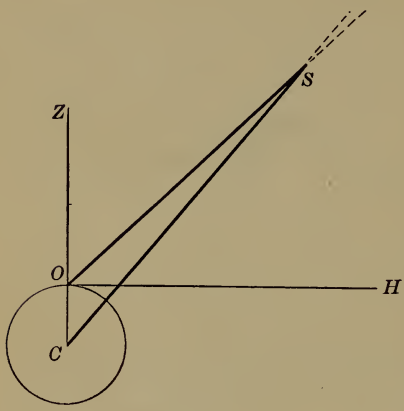


FIG. 39

Parallax may also be regarded as the angle at the celestial body made by two lines drawn from it, one to the observer and the other to the centre of the earth. Parallax therefore depends upon the *distance* of the celestial body; the nearer the celestial body the greater the parallax. The moon has the greatest parallax, then the nearest planets, and then the sun.

Let S (Fig. 39) represent the celestial body, O the position of the observer, C the centre of the earth, and Z the zenith of the observer.

The geocentric parallax is the angle

$$S = ZOS - ZCS.$$

ZOS is the apparent zenith distance, and ZCS the geocentric or true zenith distance. It is evident that geocentric parallax decreases as the altitude of the body increases.

When a body is between the horizon and zenith the parallax is called *parallax in altitude.*

Let P = the parallax in altitude,

$Z = ZOS$, the apparent zenith distance,

$R = CO$,

$D = CS$.

In the triangle COS,

$$\frac{\sin S}{CO} = \frac{\sin COS}{CS}, \quad \text{or} \quad \frac{\sin P}{R} = \frac{\sin Z}{D}.$$

Hence,
$$\sin P = \frac{R \sin Z}{D}. \qquad \text{(a)}$$

Since P is always a small angle, the arc may be taken for its sine, and the circular measure of the arc for the arc itself.

Therefore,
$$\frac{P}{57^\circ\ 17'\ 45''} = \frac{R \sin Z}{D},$$

and
$$P'' = 206265'' \frac{R \sin Z}{D}.$$

That is, parallax varies directly as the sine of the zenith distance, and inversely as the linear distance (in miles) of the body.

If the body is on the horizon as at H, the angle H is called the horizontal parallax $= P'$. In this case the angle $Z = 90^\circ$, and the preceding formula becomes

$$\sin P' = \frac{R}{D}.$$

Substituting $\sin P'$ for $\frac{R}{D}$ in formula (a), and putting h for SOH, whence $Z = 90^\circ - h$,

we have
$$\sin P = \sin P' \cos h.$$

Since P and P' are very small, their sines are nearly proportional to the angles.

Hence,
$$P = P' \cos h.$$

That is, the parallax in altitude is equal to the horizontal parallax multiplied by the cosine of the apparent altitude.

It is from this expression that the table for parallax in altitude is constructed.

The sun's mean horizontal parallax is about 8.85″, whence the sun's parallax in altitude may be computed.

This element is given in Table XVII, calculated for all altitudes of the sun from 0° to 90°. There must, of course, be separate tables for sun, moon, and planets.

As parallax causes the body to appear lower than if it were seen from the centre of the earth, the correction for parallax must evidently be *added to* the apparent altitude.

The parallax of the fixed stars is practically zero. The horizontal parallax for the planets varies, because they are at different distances from the earth at different days during the year. In the same way the horizontal parallax of the moon varies. These elements are given in the Nautical Almanac for intervals of five days. As the equatorial semidiameter of the earth is larger than any other semidiameter, so also is the equatorial horizontal parallax larger than any other horizontal parallax.

The Sun's Apparent Semidiameter

Let C (Fig. 40) represent the centre of the earth, C' the centre of the sun, and CP a tangent to the surface of the sun. Angle PCC' is the semidiameter of the sun as viewed from C.

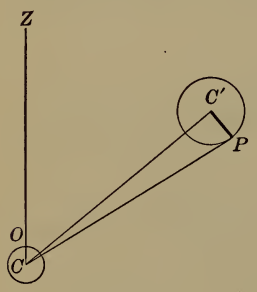


Fig. 40

Let $CC' = D$, $OC = R$, $PC' = r$, angle $PCC' = S$ = the sun's semidiameter.

In the right triangle CPC',

$$\sin S = \frac{r}{D}.$$

The formula for horizontal parallax is $\sin P' = \frac{R}{D}$, or $D = \frac{R}{\sin P'}$;

whence, $\sin S = \frac{r}{R} \sin P'$, or $S = \frac{r}{R} P'$, nearly,

since S and P' are very small.

Hence, the semidiameter of the sun may be found from its horizontal parallax.

NOTE. The semidiameter of a heavenly body when on the horizon, as viewed from the centre of the earth, is called the *horizontal semidiameter*. The semidiameter is sensibly the same whether viewed from C or O, provided the body is on the horizon, since in this position CC' and OC' are nearly equal. As the altitude of the body increases, OC' decreases, and the semidiameter as viewed from O increases. This diurnal *augmentation of the semidiameter* is sensible in the case of the moon, but not in the case of the sun. In the case of the moon there is also a monthly augmentation owing to the varying distance of the moon from the earth.

The sun's semidiameter is given for every day in the year in Table XII. It is greatest when the sun is nearest the earth, about Jan. 1, and least when the sun is farthest from the earth, about July 1. The Nautical Almanac contains the semidiameters as well as the horizontal parallaxes of the sun, moon, and planets.

In order to find the true altitude of the sun's centre, the semidiameter is added to the apparent altitude of the lower limb and subtracted from the apparent altitude of the upper limb.

APPLYING THE CORRECTIONS

The altitude of the limb is read from the sextant.

The index error being applied, the result is the *observed altitude* above the sea horizon.

The correction for dip being applied, the result is the *apparent altitude* of the limb, referred to the true horizon.

The *true altitude* of the body is the altitude of its centre as viewed from the centre of the earth, and is obtained by applying the corrections for refraction, parallax, and semidiameter to the apparent altitude of the limb.

The fixed stars have neither parallax nor semidiameter.

The following symbols will be used in this chapter:

$\odot$ the sun,
$\ominus$ the sun's centre,
$\overline{\odot}$ the sun's upper limb,
$\underline{\odot}$ the sun's lower limb.

EXAMPLE 1. The observed altitude of a star was 18° 20′ 30″; index error, + 1′ 17″; height of eye, 18 feet. Find the true altitude.

SOLUTION I

Reading of sextant	= 18° 20′ 30″
Index error	= + 1′ 17″
Observed altitude	= 18° 21′ 47″
Dip (Table XV)	= − 4′ 9″
Apparent altitude	= 18° 17′ 38″
Refraction (Table XIII)	= − 2′ 54″
True altitude	= 18° 14′ 44″

SOLUTION II

Reading of sextant	= 18° 20′ 30″	Index error	= + 1′ 17″
Correction	= − 5′ 46″	Dip	= − 4′ 9″
		Refraction	= − 2′ 54″
True altitude	= 18° 14′ 44″	Correction	= − 5′ 46″

The corrections are properly made in a certain order in the first solution; in the second solution the algebraic sum of the corrections is applied to the observed altitude. It is evident that the correction for refraction might differ by the two methods, since in one case it is based on an altitude of 18° 17′ 38″, and in the other on an altitude of 18° 20′ 30″; but in any case the difference would be so slight as to be practically unimportant in sea observations. The second method is commonly used in practice.

EXAMPLE 2. Aug. 9, 1895, the mean of observed altitudes of the sun's lower limb was 26° 0′ 2″; index error, − 0′ 17″; height of eye, 18 feet. Find the sun's true altitude.

SOLUTION I

Reading of sextant ☉	= 26° 0′ 2″
Index error	= − 0′ 17″
Observed altitude ☉	= 25° 59′ 45″
Dip	= − 4′ 9″
Apparent altitude ☉	= 25° 55′ 36″
Refraction − 1′ 59″ Parallax + 0′ 8″	= − 1′ 51″
True altitude ☉	= 25° 53′ 45″
Semidiameter (Table XII)	= + 15′ 49″
True altitude ⊖	= 26° 9′ 34″

SOLUTION II

Reading of sextant ☉	= 26° 0′ 2″	Semidiameter	= + 15′ 49″
		Parallax	= + 0′ 8″
Correction	= 9′ 32″	Index error	= − 0′ 17″
		Dip	= − 4′ 9″
		Refraction	= − 1′ 59″
True altitude ⊖	= 26° 39′ 4″	Correction	= + 9′ 32″

Do not interpolate for refraction, but take the nearest number from the table.

EXERCISE IX

From the following data compute the true altitude:

1. Obs. alt. star 25° 6′ 10″, index cor. + 1′ 15″, eye 17 feet.
2. Obs. alt. star 15° 20′ 25″, index cor. − 2′ 20″, eye 16 feet.
3. Obs. alt. ☉ 18° 17′ 30″, index cor. + 0′ 18″, eye 18 feet.
4. Obs. alt. ☉ 30° 12′ 40″, index cor. 0′ 0″, eye 19 feet.
5. Obs. alt. ☉ 56° 25′ 20″, index cor. − 1′ 20″, eye 17 feet.
6. Obs. alt. ☉ 60° 10′ 10″, index cor. − 2′ 15″, eye 18 feet.
7. Obs. alt. ☉ 31° 24′ 35″, index cor. 0′ 0″, eye 18 feet.
8. Obs. alt. ☉ 26° 17′ 20″, index cor. + 2′ 15″, eye 18 feet.
9. Obs. alt. ☉ 20° 35′ 30″, index cor. + 0′ 18″, eye 16 feet.
10. Obs. alt. ☉ 36° 12′ 10″, index cor. + 0′ 25″, eye 20 feet.

SECTION XXI

TIME

Apparent Solar Time is measured by the daily motion of the sun.

An **Apparent Solar Day** is the interval of time between two successive transits of the sun over the same meridian. This interval is not always the same, for two reasons:

(1) The sun moves in the ecliptic, which is oblique to the equator.

(2) The rate of the sun's motion in the ecliptic varies, being greatest when the earth is at perihelion, and least when the earth is at aphelion.

Hence, a clock or a chronometer cannot be regulated to follow the true sun.

The **Mean Sun** is an imaginary sun supposed to move in the equator at a uniform rate equal to the mean rate of the true sun.

Mean Solar Time is the hour angle of the mean sun, because it is the number of hours from the meridian or noon.

Apparent Solar Time is the hour angle of the true sun.

The **Equation of Time** is the difference between mean time and apparent time. It never exceeds about 16 minutes.

Sidereal Time is measured by the apparent daily motion of the fixed stars. The sidereal time at any moment may be defined as *the hour angle of the vernal equinox.*

The **Sidereal Day** is the interval between two successive transits of the vernal equinox over the same meridian. It is about 3 minutes 56 seconds shorter than the mean solar day, and is divided into 24 hours numbered from 0 to 24.

The **Civil Day** is the interval of time between two successive midnights. It is divided into two periods of 12 hours each, the first of which is marked A.M. and the second P.M.

The **Astronomical Day** begins at noon of the civil day of the same date; that is, 12 hours *after* the commencement of the civil day. It is divided into 24 hours numbered from 0 to 24.

To convert civil time into astronomical time:

If the given time is A.M., deduct 1 day, add 12 hours, and omit the A.M.; if the given time is P.M., simply omit the P.M.

To convert astronomical time into civil time:

If the hours of the given time are less than 12, simply affix P.M.; if the given hours are 12 or more, add 1 day, subtract 12 hours, and affix A.M.

EXERCISE X

	Astronomical Time	Civil Time
	d. h. m. s.	
1	July 8 7 6 10	Required
2	Mar. 7 12 25 30	Required
3	Jan. 1 18 10 10	Required
4	Dec. 31 15 0 0	Required
5	Feb. 2 8 4 30	Required
		d. h. m. s.
6	Required	July 1 11 8 25 A.M.
7	Required	Mar. 2 11 56 56 P.M.
8	Required	Aug. 31 10 8 20 P.M.
9	Required	Sept. 1 0 12 15 A.M.
10	Required	Jan. 1 10 41 56 A.M.

SECTION XXII

LONGITUDE AND TIME

Since the earth revolves on its axis once in 24 hours, the sun appears to move over the 360° of the equator in the same time. Hence, the longitude of any place is proportional to

the time required by the sun to move over an arc of the equator included between the meridian from which longitude is reckoned and the meridian of the place; and longitude may be expressed either in *time* or *arc*.

Since $360° = 24$ hours,

$$\left.\begin{aligned} 15° &= 1 \text{ hour}, \\ 15' &= 1 \text{ minute}, \\ 15'' &= 1 \text{ second}, \end{aligned}\right\} \text{ or } \left\{\begin{aligned} 1° &= 4 \text{ minutes}, \\ 1' &= 4 \text{ seconds}, \\ 1'' &= \tfrac{1}{15} \text{ second}. \end{aligned}\right.$$

Either of the above tables enables us to convert longitude in time to longitude in arc, and *vice versa*.

SECTION XXIII

THE GREENWICH DATE

The Greenwich date is the time at Greenwich corresponding to the local time elsewhere. Greenwich time should always be expressed astronomically.

To find the Greenwich date, having given the ship time and the longitude of the ship:

(1) Express the ship time astronomically (Sect. XXI, p. 389).

(2) Convert longitude in arc into longitude in time (Sect. XXII, p. 390).

(3) If the ship is in west longitude, add the longitude in time; if the ship is in east longitude, subtract the longitude in time.

The resulting Greenwich date is expressed in apparent time or mean time according as the local time is apparent or mean.

It is evident that local time can be found from Greenwich time by reversing these steps.

EXAMPLE 1. June 4, at 6 hours 10 minutes P.M., apparent time, at ship in longitude 30° 15′ E. Find the Greenwich date.

	d.	h.	m.
Ship date (expressed astronomically)	= June 4	6	10
Longitude in time	=	−2	1
Greenwich date (apparent time)	= June 4	4	9

15)30° 15′ = longitude in arc.
2 h. 1 m. = longitude in time.

EXAMPLE 2. Aug. 21, at 8 hours 30 minutes 10 seconds A.M., mean time, at ship in longitude 90° 30′ W. Find the Greenwich date.

	d.	h.	m.	s.
Ship date (expressed astronomically)	= Aug. 20	20	30	10
Longitude in time	=	+6	2	0
Greenwich date (mean time)	= Aug. 21	2	32	10

EXAMPLE 3. March 20, the sun on the meridian of the place in longitude 160° 15′ W. Find the Greenwich date.

	d.	h.	m.	s.
Ship date (expressed astronomically)	= Mar. 20	0	0	0
Longitude in time	=	+10	41	0
Greenwich date (apparent time)	= Mar. 20	10	41	0

EXERCISE XI

	Local Civil Date	Longitude	Greenwich Date
1	May 4 6 12 15 P.M.	170° 50′ 0″ W.	Required
2	July 31 11 12 30 A.M.	40° 20′ 0″ W.	Required
3	Aug. 1 2 10 15 A.M.	80° 40′ 45″ W.	Required
4	Mar. 2 10 20 0 P.M.	50° 45′ 0″ E.	Required
5	Mar. 25 11 8 0 P.M.	100° 25′ 30″ W.	Required
			d. h. m. s.
6	Required	25° 7′ 0″ W.	Dec. 30 19 47 28
7	Required	179° 0′ 0″ W.	July 4 23 51 0
8	Required	179° 0′ 0″ E.	July 3 23 59 0
9	Required	45′ 0″ E.	May 19 19 40 20 1903
10	Required	2° 10′ 0″ E.	Dec. 31 15 8 0

SECTION XXIV

THE CHRONOMETER

A chronometer is a large watch of peculiar construction, made especially for the use of navigators. It is suspended by gimbals, as a compass bowl is, and the object is to keep the face of the chronometer horizontal. It is kept in a box in the navigator's cabin and preserved as much as possible from changes of temperature and jars, for upon it depends the means of finding the longitude.

Unless otherwise specified, chronometers are supposed to be set to Greenwich mean time.

The error of chronometer on mean time at any place is the difference between the mean time at that place and the time indicated by the chronometer.

The Error of Chronometer on mean time at Greenwich is the difference between mean time at Greenwich and the time by chronometer. The error is *fast* or *slow* according as the chronometer is in advance of or behind the mean time at Greenwich.

The **Rate of Chronometer** is the daily change in its error; it is *gaining* or *losing* according as the chronometer is going too fast or too slow.

Every chronometer has an error which is accurately determined, generally at an observatory, where a memorandum is kept of its career as a timekeeper. The purchaser always receives a certificate stating how much the chronometer was too fast or too slow for Greenwich mean time at a specified date; also how much it gains or loses on the average in a day, that is, its *daily rate.* As there can be no guarantee that the rate will not change, it is important to ascertain the rate at subsequent intervals.

To determine the error of chronometer on Greenwich mean time the longitude of the place must be known, whence the

Greenwich mean time may be found (Sect. XXIII, p. 390); this compared with the chronometer set to Greenwich time gives the error.

The rate may be found by determining the error on different days and noting the daily change, supposing it to be uniform.

EXAMPLE 1. Feb. 12 the error of chronometer was 9 minutes 19.6 seconds slow, and Feb. 25 the error was 10 minutes 30.2 seconds slow. Find the rate.

		m.	s.
Feb. 12, slow	=	9	19.6
Feb. 25, slow	=	10	30.2
Change for 13 days	=	1	10.6
		60	
	13)	70.6	
Change for 1 day, or rate	=	5.4 s. losing.	

Since the chronometer was slow, and the error increasing, the rate was losing.

EXAMPLE 2. April 8, P.M., an observation was made, when the time was April 8, 4 hours 10 minutes 22 seconds by chronometer, which was 18 minutes 10 seconds fast on Greenwich mean time on Jan. 1, and 16 minutes 17 seconds fast on Jan. 31. Find the Greenwich date by chronometer.

		m.	s.
Jan. 1, fast	=	18	10
Jan. 31, fast	=	16	17
Change for 30 days	=	1	53
		60	
	30)	113	
Rate	=	3.8 s. losing.	

Since the chronometer was fast, and the error decreasing, the rate was losing.

		d.	h.	m.	s.		d.	h.	s.	
Time by chronometer	= April	8	4	10	22	Feb.	28		3.8	
Original error	=			− 16	17	March	31		228.0	60
		8	3	54	5	April	8	4	26.6	7
Accumulated rate				+ 4	15.3		67	4	0.38	0.1
Greenwich date	= April	8	3	58	20.3		67.17		0.27	0.07
255.25 s. = 4 m. 15.3 s.									255.25	

The error on Jan. 31 is 16 m. 17 s. fast, which we deduct. From Jan. 31 to April 8 d. 3 h. 58 m. is nearly 67 d. 4 h. = 67.17 d. The error for one day is 3.8 s.; hence, for 67.17 d. the error is 67.17 × 3.8 s. = 255.3 s. = 4 m. 15.3 s. As the chronometer is losing time, this accumulated rate (4 m. 15.3 s.) must be added, and the result is the required Greenwich date.

3.8 s.	
228.0	= 3.8 × 60
26.6	= 3.8 × 7
0.38	= 3.8 × 0.1
0.27	= 3.8 × 0.07
255.25	= 3.8 × 67.17

The method of multiplying 3.8 s. by 67.17 employed on page 393 will be made plain by the arrangement of the work shown in the margin.

SECTION XXV

THE NAUTICAL ALMANAC

The American and British Nautical Almanacs give, at equidistant instants of Greenwich time, the apparent right ascensions and declinations of the sun, moon, planets, and certain fixed stars; the semidiameters of the sun and moon; the equation of time; and so on.

Table XII contains the data from the Nautical Almanac which are necessary for the solution of the exercises in this chapter.

To reduce the sun's declination by means of the hourly difference:

Table XII gives the sun's declination at apparent and mean noon, Greenwich time, and the hourly change in declination for every day in the year.

If the declination at noon is required, it may be taken at once from the table. The declination at any other time may be found by simple interpolation; that is, by assuming that the declination varies as the time. The column headed "Diff. for 1 hour" contains the average change for one hour. This multiplied by the number of hours gives the correction required.

NOTE. In Table XII the difference of declination for 1 hour is marked + or − according as the sun is moving northward or southward, but in the following examples "Diff. for 1 hour" is marked + or − according as the element to which it is to be applied is increasing or decreasing numerically.

EXAMPLE 1. Find the sun's declination for 1895, Jan. 7 days 10 hours, apparent time, Greenwich date.

⊙'s dec. Jan. 7 d. 0 h. = 22° 22′ 25.8″ S. Diff. for 1 h. = − 19.23″
− 3′ 12.3″ × 10
⊙'s dec. Jan. 7 d. 10 h. = 22° 19′ 13.5″ S. Diff. for 10 h. = − 192.3″
= − 3′ 12.3″

The table gives the declination for Jan. 7 d. 0 h. = 22° 22′ 25.8″ S., and the hourly difference = 19.23″, which is marked − because the declination is decreasing. For 10 h. the declination is 10 × 19.23″ = 192.3″ = 3′ 12.3″, which must be subtracted from the declination at noon.

EXAMPLE 2. Find the sun's declination for 1895, Feb. 12 days 17 hours 12 minutes 10 seconds, mean time, Greenwich date.

⊙'s dec. Feb. 13 d. 0 h. = 13° 20′ 27.0″ S. Diff. for 1 h. = − 50.49″
+ 5′ 43.3″ × 6.8
⊙'s dec. Feb. 12 d. 17 h. Diff. for 6.8 h. = 343.332″
12 m. 10 s. = 13° 26′ 10.3″ S. = 5′ 43.3″

In this example the given date is nearer Feb. 13 d. 0 h. than Feb. 12 d. 0 h.; hence, we take from the table the declination for the former date, Feb. 13 d. 0 h.

Feb. 12 d. 17 h. 12 m. 10 s. = 6 h. 47 m. 50 s. (= 6.8 h.) before noon Feb. 13. The difference is found to be 5′ 43.3″, which must be added to the declination at noon.

EXAMPLE 3. 1895, May 5, apparent noon, longitude 57° W. Required the sun's declination.

	d.	h.	m.
Local time	= May 5	0	0
Longitude in time	=	0	3 48
Greenwich date	= May 5	3	48

⊙'s dec. May 5 d. 0 h. = 16° 15′ 42.5″ Diff. for 1 h. = + 42.76″
+ 2′ 42.5″ × 3.8
⊙'s dec. May 5 d. 3 h. 48 m. = 16° 18′ 25″ Diff. for 3.8 h. = 162.488″
= + 2′ 42.5″

To reduce the Equation of Time

Example 4. Greenwich date, mean time, 1895, March 2 days 3 hours 10 minutes. Find the equation of time.

	m.	s.		s.
Eq. of time March 2 d. 0 h.	$= -$ 12	19.64	Diff. for 1 h.	$= -$ 0.514
	$-$ 1.63			$\times$ 3.1$\frac{1}{2}$
Eq. of time March 2 d. 3 h. 10 m.	$= -$ 12	18.01	Diff. for 3.1$\frac{1}{2}$ h.	$= -$ 1.6277

Table XII gives the equation of time corresponding to mean time, March 2 d. 0 h. = 12 m. 19.64 s. The $-$ sign indicates that the equation of time is to be subtracted from mean time to obtain apparent time.

The difference for 1 hour is also found in the table and marked $-$, because the equation of time is decreasing. The difference for 3 h. 10 m. is found to be 1.63 s., which is subtracted from the equation of time at noon.

This method is not strictly correct, because the change in declination is not uniform ; but it will be found accurate enough in most cases, and will be used in this chapter.

EXERCISE XII

Find the sun's declination and the equation of time corresponding to the following Greenwich dates:

		d.	h.	m.	s.	
1.	1895 Jan.	7	3	0	0	apparent time.
2.	1895 Aug.	1	6	12	20	apparent time.
3.	1895 May	5	10	25	0	apparent time.
4.	1895 Aug.	7	15	12	0	apparent time.
5.	1895 Dec.	4	6	18	0	apparent time.
6.	1895 July	23	20	16	40	apparent time.
7.	1895 Nov.	1	3	6	0	apparent time.
8.	1895 Oct.	12	5	12	0	apparent time.
9.	1895 June	7	3	18	0	mean time.
10.	1895 Feb.	3	9	15	0	mean time.

SECTION XXVI

TO FIND THE HOUR ANGLE OF A HEAVENLY BODY

Let t = the hour angle of the heavenly body.
L = the latitude of the observer.
d = the declination of the heavenly body.
h = the true altitude of the heavenly body.
$p = 90° \pm d$
= the polar distance of the heavenly body.
$S = \frac{1}{2}(L + p + h)$.
$R = \frac{1}{2}(L + p - h)$
$= \frac{1}{2}(L + p + h) - h$
$= S - h$.

The general formula for the hour angle of a heavenly body in terms of its polar distance and altitude and the latitude of the observer is

$$\sin \tfrac{1}{2}\,\mathrm{t} = \pm\,[\cos \mathrm{S} \sin \mathrm{R} \sec \mathrm{L} \csc \mathrm{p}]^{\frac{1}{2}}.$$

(*Spherical Trigonometry*, LXX, p. 192.)

Example 1. The true altitude of a heavenly body is 14° 50′ 42″; its declination, 16° 20′ 5″ S.; and the latitude of the observer, 37° N. Find the hour angle.

$L =$	37°	
$p =$	106° 20′ 5″	$\log \cos S = 9.27708$
$h =$	14° 50′ 42″	$\log \sin R = 9.95456$
$2\,S =$	158° 10′ 47″	$\log \sec L = 0.09765$
$S =$	79° 5′ 24″	$\log \csc p = 0.01789$
$R =$	64° 14′ 42″	$2\overline{)19.34718}$
		$\log \sin \frac{1}{2}t = 9.67359$

$\frac{1}{2}t = 28°\ 8′\ 22.5″$
$\therefore t = 56°\ 16′\ 45″$
$= 3$ h. 45 m. 7 s.

SECTION XXVII

TO FIND THE LOCAL TIME

The local apparent time is the hour angle of the true sun, if the sun is west of the meridian; if the sun is east of the meridian, the local apparent time is the difference between 24 hours and the hour angle. If, however, we count the hour angle around the circle, the sun's hour angle is the apparent time.

EXAMPLE 1. At sea, Sept. 12, A.M., in latitude 18° 12′ S., Greenwich mean time, 1895, Sept. 11 days 14 hours 55 minutes 50 seconds; observed altitude of ☉, 26° 18′ 20″; index error, + 31″; height of eye, 18 feet. Required the local mean time.

Sept. 11 d. 14 h. 55 m. 50 s. = 9 h. 4 m. 10 s. (=9.07 h.) before noon, Sept. 12.

To reduce the sun's declination and find the polar distance:

☉'s dec. Sept. 12 d. 0 h.	= 4° 10′ 47.1″ N.	Diff. for 1 h.	= − 57.32″
Change for 9.07 h.	= + 8′ 39.9″	Diff. for 9 h.	= 515.88″
☉'s dec. Sept. 11 d. 14 h.		Diff. for 0.07 h.	= 4.01″
55 m. 50 s.	= 4° 19′ 27″ N.	Diff. for 9.07 h.	= 519.89″
	90°		= 8′ 39.9″
	p = 94° 19′ 27″		

To find the true altitude:

Reading of sextant ☉	= 26° 18′ 20″	Semidiam.	= + 15′ 56″
Correction	= + 10′ 29″	Index error	= + 31″
		Parallax	= + 8″
		Dip	= − 4′ 9″
		Refraction	= − 1′ 57″
True altitude ⊖	= 26° 28′ 49″	Correction	= + 10′ 29″

To reduce the equation of time:

	m. s.		s.
Eq. of time Sept. 12 d. 0 h.	= 3 44.76	Diff. for 1 h.	= + 0.874
Change for 9.07 h.	= − 7.93	Diff. for 9 h.	= 7.866
Eq. of time Sept. 11 d. 14 h.		Diff. for 0.07 h.	= 0.061
55 m. 50 s.	= 3 36.8	Diff. for 9.07 h.	= 7.927

To find the hour angle of the sun:

$$\sin \tfrac{1}{2} t = \pm [\cos S \sin R \sec L \csc p]^{\frac{1}{2}}.$$

$L =$ 18° 12′ 0″	$\log \cos S =$ 9.54428
$p =$ 94° 19′ 27″	$\log \sin R =$ 9.83396
$h =$ 26° 28′ 49″	$\log \sec L =$ 0.02229
$2S =$ 139° 0′ 16″	$\log \csc p =$ 0.00123
$S =$ 69° 30′ 8″	2)19.40176
$R =$ 43° 1′ 19″	$\log \sin \frac{1}{2} t =$ 9.70088

$\frac{1}{2} t = 30° 8′ 46″$

$\therefore t = 60° 17′ 32″$

$= 4$ h. 1 m. 10 s.

		h.	m.	s.
		24	0	0
		4	1	10
Local apparent astronomical time	= Sept. 11,	19	58	50
Equation of time	=		− 3	37
Local mean astronomical time	= Sept. 11,	19	55	13
Local mean civil time	= Sept. 12,	7	55	13 A.M.

Example 2. 1895, Aug. 23, P.M., at sea, in latitude 9° 37′ N. Five observed altitudes of the ☉ were taken at the times (by the watch) standing opposite:

Readings of sextant ☉:	h.	m.	s.
21° 33′ 30″	2	42	58
21° 28′ 10″	2	43	21.5
21° 24′ 00″	2	43	39
21° 19′ 10″	2	43	59.5
21° 14′ 30″	2	44	18.5

Index error, 0; height of eye, 18 feet; correction of watch by chronometer, − 5 hours 16 minutes 49.3 seconds. Required the local time.

Note. Since the determination of the time at sea requires that the altitude of the object should be taken with great accuracy, a single observation for this purpose is seldom considered as sufficient; it is, therefore, usual to take a set of altitudes and to employ the mean of the

whole, taking the mean of the corresponding times by the watch as the estimated time.

The mean of the five observed altitudes is found by dividing their sum by 5, and the mean time (by a watch) of these observations is found in a similar manner.

Readings of sextant ☉.

	h.	m.	s.
21° 33′ 30″	2	42	58
21° 28′ 10″	2	43	21.5
21° 24′ 00″	2	43	39
21° 19′ 10″	2	43	59.5
21° 14′ 30″	2	44	18.5
119′ 20″	218		16.5
21° 23′ 52″	2	43	39.3

We are to proceed, therefore, as if the observed altitude of the sun's lower limb was 21° 23′ 52″ and the corresponding time per watch 2 h. 43 m. 39.3 s. P.M.

	h.	m.	s.
	2	43	39.3
−	5	16	49.3
	9	26	50 A.M.

The Greenwich mean time is found to be Aug. 22 d. 21 h. 26 m. 50 s., which is 2 h. 33 m. 10 s. before noon, Aug. 23.

Aug. 22, 21 26 50 G.M.T.
2 33 10 = 2.55 h.

☉'s dec., Aug. 23, 0 h. 11° 27′ 14.7″ N.
+ 2′ 9.9″
11° 29′ 24.6″ N.
90°
p = 78° 30′ 35″

Diff. for 1 h.	= − 50.94″
Diff. for 2 h.	= 101.88″
Diff. for 0.5 h.	= 25.47″
Diff. for 0.05 h.	= 2.55″
Diff. for 2.55 h.	= 129.90″
	= 2′ 9.9″

Semidiam.	= + 15′ 52″
Parallax	= + 8″
Dip	= − 4′ 9″
Refraction	= − 2′ 28″
Correction	= + 9′ 23″

Eq. of time. m.	s.	Diff. for 1 h. s.
2	32.07	− 0.642
	1.64	1.284 2
2	33.71	0.321 0.5
		0.032 0.05
		1.637

L =	9° 37′ 00″	log cos S =	9.76031
p =	78° 30′ 35″	log sin R =	9.73943
h =	21° 33′ 15″	log sec L =	0.00615
2 S =	109° 40′ 50″	log csc p =	0.00879
S =	54° 50′ 25″		2)19.51468
R =	33° 17′ 10″	log sin ½t =	9.75734
		½t =	34° 53′ 3″
		t =	69° 46′ 6″

h. m. s.
∴ t = 4 39 4
Equation of time = + 2 34
Local mean time = 4 41 38 P.M.

SECTION XXVIII

LATITUDE BY MERIDIAN OBSERVATION OF A HEAVENLY BODY

Latitude is the angle between the direction of gravity and the plane of the equator. Hence, to find the latitude we have to find the distance of the zenith from the equator; that is, the declination of the zenith.

Let HZH' (Fig. 41) represent the meridian of the observer, P the nearest pole; Z the zenith, HH' the horizon, EQ the equator, and S the heavenly body on the meridian.

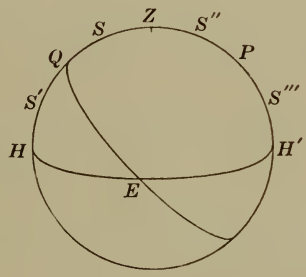


FIG. 41

$QS = d =$ the declination of the heavenly body.

$HS = h =$ the true altitude of the heavenly body.

$SZ = z = 90° - h =$ the zenith distance of the heavenly body. The zenith distance is named N. or S. according as the zenith is north or south of the heavenly body.

$PQ = ZH'$ (each being equal to 90°); taking away ZP, we have $PH' = QZ$; that is, the altitude of the pole is equal to the latitude of the observer.

$$QZ = SZ + QS, \qquad \text{or} \quad L = z + d.$$

If the heavenly body is at S',

$$QZ = S'Z - QS', \qquad \text{or} \quad L = z - d.$$

If the heavenly body is at S'',

$$QZ = QS'' - ZS'', \quad \text{or} \quad L = d - z.$$

If the heavenly body is at S''',

$$QZ = QS''' - ZS''', \quad \text{or} \quad L = d - z.$$

Hence, to find the latitude of a place by meridian observation of a heavenly body:

If the declination and zenith distance of the body are of the same name, take their arithmetical sum; if of different names, take their arithmetical difference and give it the name (N. or S.) of the greater.

That is, the algebraic sum of the declination and the zenith distance is the latitude, and the latitude is of the name of the greater.

The following method may also be used to find the latitude from the meridian altitude of a heavenly body:

(1) The meridian altitude of the sun is found to be 25° from the south point of the horizon, that is, 65° south of the zenith, and the Nautical Almanac gives its declination on that day as 15° S. Find the latitude of the observer.

SOLUTION. If the observer was in 15° S. latitude, the sun would be in the zenith; as the sun is 65° south of the zenith, the observer must be 65° to the north of 15° S., or in latitude 50° N.

(2) If the meridian altitude of the sun is 10° from the north point of the horizon, that is, 80° north of the zenith, and the Nautical Almanac gives its declination at the time as 20° N., what is the latitude of the observer?

SOLUTION. If the observer was in 20° N. latitude, the sun would be in the zenith; as the sun is 80° north of the zenith, the observer must be 80° to the south of 20° N., or in latitude 60° S.

(3) The sun's meridian altitude is 60° from the north point of the horizon, and the Nautical Almanac gives the sun's declination as 10° N. Find the latitude of the observer.

SOLUTION. If the observer was in 10° N. latitude, the sun would be in the zenith; as the sun is 90° − 60° = 30° north of the zenith, the observer must be 30° to the south of 10° N., or in latitude 20° S.

(4) The declination of a star is 34° N., and its altitude when on the meridian is 56° 27′ from the south point of the horizon. Find the latitude of the observer.

SOLUTION. If the observer was in 34° S. latitude, the star would be in the zenith; as the star when on the meridian is 90° − 56° 27′ = 33° 33′ south of the zenith, the observer must be 33° 33′ to the north of 34° N., or in latitude 67° 33′ N.

(a) *Latitude by meridian altitude of the sun.*

The common mode at sea of measuring a meridian altitude of the sun is to commence observing the altitude 20 or 30 minutes before noon, repeating the operation until the highest altitude is attained; soon after which the sun, as seen through the sight tube of the instrument, begins to *dip* or descend below the line of the horizon.

EXAMPLE. 1895, Aug. 11, in longitude 92° 12′ E., by account, the observed meridian altitude of the sun's lower limb was 42° 42′ 10″ N.; index error, − 2′ 50″; height of the eye, 17 feet. Find the latitude.

Since the sun bears *north*, the zenith distance is *south*.

		d.	h.	m.	s.
Ship date	= Aug.	11	0	0	0
Longitude 92° 12′ E.	=		6	8	48
Greenwich date	= Aug.	10	17	51	12

Reading of sextant ⊙ = 42° 42′ 10″	Index error = − 2′ 50″	⊙'s dec. Aug. 11	−44.51″
	Semidiam. = + 15′ 50″	= 15° 17′ 7″ N.	× 6.15
	Dip = − 4′ 2″	+ 4′ 34″	273.74″
Cor. = + 8′ 2″	Refraction = − 1′ 3″	d = 15° 21′ 41″ N.	= 4′ 34″
	Parallax = + 7″	z = 47° 9′ 48″ S.	
True alt. ⊖ = 42° 50′ 12″	Correction = + 8′ 2″	L = 31° 48′ 7″ S.	
90°			
z = 47° 9′ 48″ S.			

EXERCISE XIII

From the following data find the latitude:

	Civil Date	Longitude	Obs. Merid. Alt. ☉	Index Cor.	Eye
1	1895, Jan. 1	102° 41′ W.	59° 59′ 50″ S.	+ 0′ 50″	15
2	1895, Feb. 1	78° 14′ E.	78° 4′ 10″ S.	+ 0′ 55″	12
3	1895, Mar. 20	173° 18′ W.	89° 37′ 0″ N.	+ 4′ 32″	18
4	1895, April 1	87° 42′ W.	48° 42′ 30″ S.	+ 1′ 42″	18
5	1895, Sept. 1	97° 42′ E.	51° 4′ 50″ S.	− 6′ 0″	23
6	1895, Aug. 26	92° 3′ E.	35° 35′ 20″ N.	+ 2′ 17″	12
7	1895, May 16	45° 26′ W.	86° 34′ 20″ N.	+ 4′ 16″	15
8	1895, Mar. 20	174° 0′ W.	89° 56′ 10″ N.	− 1′ 15″	15
9	1895, June 1	44° 40′ E.	72° 14′ 10″ N.	+ 3′ 45″	22
10	1895, Dec. 1	67° 56′ E.	18° 48′ 10″ S.	− 3′ 6″	18
11	1895, Sept. 23	57° 45′ E.	84° 10′ 50″ N.	− 1′ 36″	16
12	1895, Sept. 23	119° 54′ E.	83° 46′ 0″ S.	− 5′ 30″	18
13	1895, Nov. 21	70° 20′ E.	80° 20′ 0″ N.	− 2′ 50″	20
14	1895, Dec. 31	123° 45′ W.	67° 8′ 10″ S.	+ 0′ 9″	13
15	1895, Oct. 20	150° 25′ W.	49° 58′ 50″ N.	+ 1′ 10″	19
16	1895, June 1	96° 17′ E.	75° 38′ 15″ N.	+ 0′ 27″	26
			Obs. Merid. alt. ☉		
17	1895, June 25	59° 15′ E.	60° 23′ 15″ N.	+ 2′ 21″	30

(b) *Latitude by meridian altitude of a fixed star.*

Since the declination of a fixed star changes very slowly, it is sensibly the same for several successive days; hence, the Greenwich date and the longitude are not required.

Example. 1895, Dec. 24, the observed meridian altitude of the star Procyon (declination 5° 29′ 34″ N.) was 52° 51′ 50″ N.; index error, − 49″; height of eye, 21 feet. Find the latitude.

Reading of sextant	= 52° 51′ 50″ N.	Index error	= − 0′ 49″
Correction	= − 6′ 2″	Dip	= − 4′ 29″
True altitude	= 52° 45′ 48″ N.	Refraction	= − 0′ 44″
	90°	Correction	= − 6′ 2″
Zenith distance	= 37° 14′ 12″ S.		
Declination	= 5° 29′ 34″ N.		
Latitude	= 31° 44′ 38″ S.		

Declinations of Fixed Stars

Date	Star	Declination
1895, Jan. 29	Aldebaran	16° 18′ 2″ N.
1895, Feb. 8	Sirius	16° 34′ 20″ S.
1895, Feb. 18	Procyon	5° 29′ 39″ N.
1895, Mar. 20	Arcturus	19° 43′ 23″ N.
1895, Mar. 30	Spica	10° 37′ 4″ S.
1895, April 9	Sirius	16° 34′ 24″ S.
1895, July 8	Antares	26° 12′ 12″ S.
1895, Aug. 17	Altair	8° 35′ 34″ N.
1895, Aug. 17	β Centauri	59° 52′ 29″ S.
1895, Sept. 6	Arcturus	19° 43′ 37″ N.
1895, Oct. 6	Markab	14° 38′ 49″ N.
1895, Nov. 5	Fomalhaut	30° 10′ 32″ S.
1895, Dec. 4	α Arietis	22° 58′ 26″ N.
1895, Dec. 24	Procyon	5° 29′ 34″ N.

EXERCISE XIV

Using the declinations of fixed stars given in the above table, find the latitude:

	Civil Date	Star	Obs. Merid. Alt.	Index Cor.	Eye
1	1895, Jan. 29	Aldebaran	52° 36′ 0″ S.	− 0′ 23″	20
2	1895, Feb. 18	Procyon	77° 18′ 10″ S.	+ 0′ 19″	16
3	1895, Mar. 20	Arcturus	36° 10′ 20″ N.	+ 2′ 42″	20
4	1895, Aug. 17	Altair	66° 51′ 10″ N.	+ 0′ 58″	13
5	1895, Nov. 5	Fomalhaut	59° 40′ 0″ N.	+ 1′ 12″	23
6	1895, Sept. 6	Arcturus	86° 35′ 50″ N.	− 1′ 10″	12
7	1895, Oct. 6	Markab	54° 10′ 15″ S.	0	13
8	1895, Aug. 17	β Centauri	59° 47′ 13″ S.	0	25
9	1895, Dec. 4	α Arietis	60° 29′ 50″ S.	− 2′ 10″	18
10	1895, Feb. 8	Sirius	37° 50′ 20″ S.	+ 1′ 4″	19
11	1895, April 9	Sirius	61° 3′ 50″ N.	0	16
12	1895, Mar. 30	Spica	52° 14′ 0″ N.	0	19
13	1895, July 8	Antares	70° 10′ 30″ N.	0	21

SECTION XXIX

LATITUDE BY EX-MERIDIAN ALTITUDE OF THE SUN BY REDUCTION TO THE MERIDIAN

If it happens that the meridian altitude of the sun cannot be observed owing to clouds or other causes, the altitude may be observed before or after noon and reduced to the meridian altitude.

Towson's Method

Towson's Tables for the reduction of ex-meridian altitudes furnish a simple practical solution of the problem under consideration.

Corresponding to the nearest hour angle (Sect. XXVI, p. 397) and nearest declination (Sect. XXV, p. 394) of the sun at the time of observation, Towson's first table gives an *Index Number* and *Augmentation I.* The *declination of the sun at noon* is found by adding Augmentation I to the declination at the time of observation.

Corresponding to the Index Number (given by the first table) and the nearest true altitude of the sun at the time of observation, Towson's second table gives Augmentation II. The *true altitude of the sun at noon* is found by adding Augmentation II to the true altitude at the time of observation.

The declination and the altitude at noon being known, the latitude is found as in Sect. XXVIII, p. 401.

The latitude may be found from these tables by ex-meridian altitudes of a star or planet, but sidereal time must be employed in finding the hour angle.

If equal altitudes are observed before and after meridian passage, the hour angle may be found by taking half of the elapsed time; or, if the two altitudes differ only a few minutes (of arc), their mean may be reduced by employing half the elapsed time as the hour angle.

It should be observed that Towson's method is independent of the latitude by account.

The maximum altitude in Towson's Tables is 74°, and the hour angle corresponding cannot exceed 22 minutes. The greatest declination is 23° 20′.

Towson's method will be illustrated by a single example.

EXAMPLE. 1895, Dec. 2, P.M., at ship, in longitude 4° 39′ W.; observed altitude ⊙ south of observer was 23° 14′ 10″; height of eye, 11 feet; time by watch, Dec. 2, 0 hours 50 minutes 58 seconds, which had been found to be 19 minutes 38 seconds fast on apparent time at ship; the difference of longitude made to the west was 21.3′ after the error on apparent time was determined. Required the latitude.

		d. h. m. s.		d. h. m. s.
Time by watch	= Dec.	2 0 50 58	App. time at ship	= Dec. 2 0 29 55
Watch fast	=	− 19 38	Longitude, 4° 39′ W. =	+ 16 36
		2 0 31 20	Greenwich date	= Dec. 2 0 46 31
Diff. long. $\frac{21.3}{15}$	=	− 1 25	46 m. 31 s. = 0.78 h.	
App. time at ship	= Dec.	2 0 29 55	⊙'s dec., Dec. 2 d. 0 h.	
Time from noon	=	29 55	= 21° 58′ 41.3″	22.23″
			17.3″	0.78
			Reduced declination = 21° 58′ 59″	17.34″
			Aug. (Towson, Table I) = + 10′ 17″	
			Meridian declination = 22° 9′ 16″	

Reading of sextant ⊙	= 23° 14′ 10″		
Correction	= + 10′ 54″	Semidiameter	= + 16′ 16″
		Dip	= − 3′ 15″
		Parallax	= + 0′ 8″
		Refraction	= − 2′ 15″
True altitude	= 23° 25′ 4″	Correction	= + 10′ 54″
Aug. (Towson, Table II)	= + 11′ 2″		
Meridian altitude	= 23° 36′ 6″		
	90°		
Zenith distance	= 66° 23′ 54″		
Meridian declination	= 22° 9′ 16″		
Latitude	= 44° 14′ 38″ N.		

SECTION XXX

LONGITUDE BY CHRONOMETER FROM AN OBSERVED ALTITUDE OF THE SUN

The difference of longitude of two places is equal to the difference of time reduced to arc. (Sect. XXII, p. 389.)

Hence, the longitude of a place from Greenwich may be determined by taking the difference between the local mean time and the Greenwich mean time and reducing to arc. The longitude will be east or west according as the Greenwich time is less or greater than local time.

The local mean time is found as in Sect. XXVII, p. 398. The Greenwich mean time is found by chronometer set to Greenwich time. (See Sect. XXIV, p. 392, also the example of Sect. XXIX, p. 407.)

As an astronomical question the determination of longitude resolves itself into the determination of the difference of time *reckoned at the two meridians at the same* ABSOLUTE *instant.*

If we *could* note the exact moment the sun is on the meridian, it would then be 12 o'clock *apparent* local time, and by applying the equation of time to it we should have the *mean* local time. The difference between it and the time shown by the chronometer at the moment of observation would be the difference of longitude in time. But we cannot determine the *instant* that the sun attains his meridian altitude. It is true that in the tropics, where the sun has always a high meridian altitude and where its motion in altitude is sufficiently rapid for a good observation at any point of his course, we can mark it pretty clearly; but in other latitudes, especially when the sun's meridian altitude is low, we cannot mark it within three or four minutes, and a minute of time represents 15′ of longitude. Navigators never rely on this method to determine longitude.

The longitude of any place on the surface of the earth is ascertained as soon as we can discover the time at that place and the time at Greenwich at the same instant, since we have only to change the difference of the two times into degrees by multiplying by 15.

In the morning or evening, when the sun is 15° or 20° high and is rising or falling rapidly, the navigator observes its altitude, and notes the time by chronometer. He then solves a spherical triangle, of which he has given the three sides, the altitude just observed, the latitude, and the sun's polar distance. The latitude is really found at noon, as previously explained, and is "worked back" to the time of taking the morning observation. The polar distance of the sun is 90° ± its declination, which declination is found in the Nautical Almanac.

EXAMPLE 1. 1895, Feb. 10, A.M., at ship, latitude 50° 48′ N.; observed altitude ☉, 9° 10′ 50″; index correction, − 3′ 18″; height of eye, 18 feet; time by chronometer, Feb. 9 days 9 hours 59 minutes 25 seconds, which was 37 minutes 58.8 seconds fast for mean noon at Greenwich, Dec. 20, 1894, and on Jan. 10, 1895, was 34 minutes 12 seconds fast for mean noon at Greenwich; from the time of observation until noon the ship has sailed on a S.W. course (true) 34 miles. Required the longitude at noon.

At sea the latitude *by account* is used, either brought forward from a preceding determination or carried backward from a subsequent determination.

	m.	s.
Dec. 20, chronometer fast =	37	58.8
Jan. 10, chronometer fast =	34	12.0
Change in 21 days =	3	46.8
	60	
	226.8	

21 { 3)226.8
7)75.6
10.8 losing
324.0 30
4.3 0.4

60)328.3

5′ 28.3″

The change in 1 day is 226.8 seconds ÷ 21 = 10.8 seconds; the change in 30.4 days (the interval between noon Jan. 10 and Feb. 9 days 10 hours) is 30.4 × 10.8 seconds = 328.3 seconds = 5 minutes 28.3 seconds.

		d.	h.	m.	s.
Time by chronometer	= Feb.	9	9	59	25
Original error	=			− 34	12
		9	9	25	13
Accumulated rate	=			+ 5	28
Greenwich mean time	=	9	9	30	41

Time, after noon 9 h. 30 m. 41 s. = 9.51 h.

⊙'s declination = 14° 39′ 24.8″ S.
− 7′ 38.1″
14° 31′ 46.7″
90°
p = 104° 31′ 46.7″

+ 48.17″
× 9.51
458.10″
7′ 38.1″

Reading of sextant ⊙ = 9° 10′ 50″
Correction = + 3′ 10″ { Semidiam. = +16′ 15″ ; Parallax = + 9″

Ind. er. = − 3′ 18″
Dip = − 4′ 9″
Ref. = − 5′ 47″
Cor. = + 3′ 10″

True alt. ⊖ = 9° 14′ 0″

L = 50° 48′ 0″
p = 104° 31′ 47″
h = 9° 14′ 0″
2 S = 164° 33′ 47″
S = 82° 16′ 53″
R = 73° 2′ 53″

log cos S = 9.12810
log sin R = 9.98071
log sec L = 0.19926
log csc p = 0.01411
2) 19.32218
log sin ½ t = 9.66109
∴ ½ t = 27° 16′ 24″
t = 54° 32′ 48″
= 3 h. 38 m. 11 s. Time before noon, Feb. 10.

Eq. of time.

m.	s.
14	25.05
+	0.53
14	25.58

+ 0.056
× 9.51
0.533

	d.	h.	m.	s.	
Feb.	9	20	21	49	local apparent astronomical time.
		+14	26		equation of time.
Feb.	9	20	36	15	local mean astronomical time.
Feb.	9	9	30	41	Greenwich mean time.
	11	5	34		difference of time.

166° 23′ 30″ E. long. at time of observation.

Longitude at sights = 166° 23′ E.
Change until noon = 38′ W.
Longitude at noon = 165° 45′ E.

Latitude at sights = 50° 48′ N.
Change until noon = 24′ S.
Latitude at noon = 50° 24′ N.

From the time of observation until noon the ship sailed S. W. 34 miles. The difference of latitude and the difference of longitude corresponding to this course and distance are found by Middle Latitude Sailing to be 24′ S. and 38′ W.

At sea it is usual to reduce longitudes obtained from day observations to noon by allowing for the run of the ship in the interval.

EXAMPLE 2. 1895, Nov. 8, P.M., at sea, in latitude 30′ S.; five observed altitudes of the ☉ were taken at the times (by watch) standing opposite:

		h.	m.	s.
Readings of sextant ☉:	24° 51′ 50″	11	8	3
	48′ 50″			16
	44′ 40″			33
	40′ 20″			49
	37′ 0″			64.5

Index error, + 30″; height of eye, 18 feet; correction of watch by chronometer, − 5 hours 12 minutes 15.9 seconds. Required the longitude.

	h.	m.	s.
24° 51′ 50″	11	8	3
48′ 50″			16
44′ 40″			33
40′ 20″			49
37′ 0″			64.5
222′ 40″			165.5
24° 44′ 32″	11	8	33.1
	− 5	12	15.9
	5	56	17.2

= 5.94 h. P.M.

☉'s declination
16° 35′ 43.2″ S.
+ 4′ 19.0″
16° 40′ 2.2″ S.
90°
p = 73° 19′ 58″

− 43.60″
× 5.94
258.98″
= 4′ 19.0″

Semidiameter	= + 16′ 11″
Parallax	= + 8″
Index error	= + 30″
Dip	= − 4′ 9″
Refraction	= − 2′ 6″
Correction	= + 10′ 34″

Reading of sextant ☉ = 24° 44′ 32″
Correction = + 10′ 34″
True altitude ⊖ = 24° 55′ 6″

L = 0° 30′ 0″	log cos S = 9.81365
n = 73° 19′ 58″	log sin R = 9.61701
h = 24° 55′ 6″	log sec L = 0.00002
2 S = 98° 45′ 4″	log csc p = 0.01864
S = 49° 22′ 32″	2) 19.44932
R = 24° 27′ 26″	log sin ½ t = 9.72466

½ t = 32° 2′ 14″.
t = 64° 4′ 28″.
∴ t = 4 h. 16 m. 18 s.
Time after noon, Nov. 8.

	m.	s.		s.
Eq. of time.	16	9.10		− 0.180
	−	1.07		× 5.94
	16	8.03		1.07

d.	h.	m.	s.	
Nov. 8	4	16	18	local apparent astronomical time.
		16	8	equation of time.
Nov. 8	4	0	10	local mean astronomical time.
Nov. 8	5	56	17	Greenwich mean time.
	1	56	7	difference of time.
	29°	1′	45″	W. longitude.

Problem for the "Flying Dutchman"*

In what way could a navigator who has no knowledge of his position and has lost all record of time ascertain not only his correct latitude and longitude but recover the day and date?

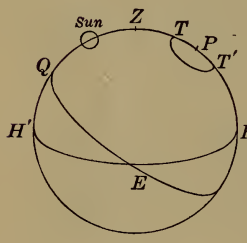


Fig. 42

Let $H'ZH$ (Fig. 42) represent the meridian, P the pole, Z the zenith, EQ the equator, TT' any circumpolar star, and $H'H$ the horizon.

(1) Observe altitude of a circumpolar star at upper and lower culmination. One half the sum of these altitudes is equal to the altitude of the pole, which in turn equals the latitude of the place.

(2) Heave-to the ship so as to keep her on the same parallel of latitude until noon of the next day.

(3) At noon observe the meridian altitude of the sun, which subtracted from 90° gives zenith distance.

(4) Knowing zenith distance and latitude we can find sun's declination.

(5) Look in Nautical Almanac to see on what day the sun has this declination.

* Lecky's *Wrinkles in Practical Navigation.*

NOTE. There are, however, two difficulties in carrying out these directions in actual practice:

(1) As the observer is not supposed to know whether the sun is moving north or south, it would be necessary to wait and take *two* meridian altitudes to determine on which side of the solstice the sun is.

(2) As the observer's longitude is unknown and as the elements in the Nautical Almanac are computed for noon Greenwich date and not for noon at ship, he would still be uncertain as to the exact date.

EXERCISE XV

1. 1895, Oct. 19, A.M., at sea, in latitude 33° 27′ S.; the observed altitude $\underline{\odot}$, 28° 22′ 30″; index correction, + 30″; height of eye, 18 feet; Greenwich mean time by chronometer, Oct. 18 d. 18 h. 28 m. 38 s. Required the longitude.

2. 1895, Oct. 20, A.M., at sea, in latitude 31° 40′ S.; the observed altitude $\underline{\odot}$, 35° 16′ 10″; index correction, + 30″; height of eye, 18 feet; Greenwich mean time by chronometer, Oct. 19 d. 19 h. 11 m. 24 s. Required the longitude.

3. 1895, Oct. 20, P.M., at sea, in latitude 30° 55′ S.; the observed altitude $\underline{\odot}$, 21° 42′ 30″; index correction, + 29″; height of eye, 18 feet; Greenwich mean time by chronometer, Oct. 20 d. 3 h. 35 m. 40 s. Required the longitude.

4. 1895, Oct. 21, A.M., at sea, in latitude 29° 35′ S.; the observed altitude $\underline{\odot}$, 24° 26′ 42″; index correction, + 29″; height of eye, 18 feet; Greenwich mean time by chronometer, Oct. 20 d. 18 h. 30 m. 39 s. Required the longitude.

5. 1895, Jan. 29, P.M., at ship, latitude 42° 26′ N.; observed altitude $\underline{\odot}$, 13° 40′; index correction, − 1′ 8″; height of eye, 16 feet; time by chronometer, 29 d. 6 h. 48 m. 40 s., which was slow 11 m. 22.3 s. for mean noon at Greenwich, Dec. 1, 1894, and on Jan. 1, 1895, was 8 m. 7 s. slow for Greenwich mean noon. Required the longitude.

6. 1895, March 31, A.M., at ship, latitude 26° 9′ N.; observed altitude $\underline{\odot}$, 29° 10′ 20″; height of eye, 26 feet; time by

chronometer, 31 d. 0 h. 4 m. 50 s., which was 58 m. 58 s. fast for mean noon at Greenwich, Nov. 20, 1894, and on Dec. 31, 1894, was 1 h. 2 m. 55.8 s. fast for mean time at Greenwich. Required the longitude.

7. 1895, May 22, A.M., at ship, latitude 43° 25′ N.; observed altitude ⊙, 32° 8′; index correction, + 1′ 28″; height of eye, 15 feet; time by chronometer, 21 d. 21 h. 6 m. 10 s., which was slow 12.6 s. for mean noon at Greenwich, Feb. 24, and on April 1 was 2 m. 45 s. fast for mean noon at Greenwich. Required the longitude.

8. 1895, Aug. 24, A.M., at ship, latitude at noon, 37° 59′ N.; observed altitude ⊙, 37° 13′ 30″; index correction, + 2′ 44″; height of eye, 18 feet; time by chronometer, Aug. 23 d. 18 h. 13 m. 24 s., which was 1 m. 5 s. fast for mean noon at Greenwich, Aug. 1, and on Aug. 10 was 0 m. 42 s. slow for mean time at Greenwich; course (true) since observation, N.N.W.; distance, 22.4 miles. Required the longitude at noon.

9. 1895, Jan. 29, P.M., at ship, latitude 28° 45′ N.; observed altitude ⊙, 17° 46′ 30″; index correction, − 3′ 18″; height of eye, 16 feet; time by chronometer, Jan. 28 d. 16 h. 31 m. 30 s., which was 1 m. 16.5 s. fast for Greenwich mean noon, Dec. 17, 1894, and on Jan. 1, 1895, was 1 m. 3 s. slow for Greenwich mean time; course (true) since noon, N.W. by W.; distance, 20 miles. Required the longitude at the time of observation, and also at noon.

10. 1895, Aug. 31, P.M., at ship, latitude 0°; observed altitude ⊙, 45° 5′ 30″; index correction, − 2′ 4″; height of eye, 15 feet; time by chronometer, Aug. 31 d. 9 h. 11 m. 28 s., which was 5 m. 20 s. fast for mean noon at Greenwich, April 15, and on June 16 was fast 2 m. 43 s. on mean time at Greenwich. Required the longitude.

11. 1895, April 15, A.M., at ship, latitude 48° 52′ N.; observed altitude ⊙, 22° 18′; index correction, − 3′ 54″;

height of eye, 17 feet; time by chronometer, April 14 d. 22 h. 30 m. 42 s., which was 0 m. 4 s. slow for mean noon at Greenwich, Jan. 1, and on Jan. 12 was fast 0 m. 2 s. Required the longitude.

12. 1895, Aug. 28, P.M., at ship, latitude 5° S.; observed altitude $\underline{\odot}$, 38°; index correction, + 5′ 27″; height of eye, 21 feet; time by chronometer, Aug. 27 d. 22 h. 20 m. 30 s., which was 10 m. 0 s. slow for mean noon at Greenwich, Feb. 19, and on May 30 was 2 m. 20 s. slow on mean noon at Greenwich. Required the longitude.

13. 1895, Sept. 22, A.M., at ship, on the equator, observed altitude $\overline{\odot}$, 17° 20′ 40″; index correction, −1′ 9″; height of eye, 20 feet; time by chronometer, Sept. 22 d. 4 h. 59 m. 16 s., which was 15 s. slow for Greenwich mean noon, April 30, and on June 1 was 10.6 s. fast for mean time at Greenwich. Required the longitude.

14. 1895, Aug. 5, A.M., at ship, latitude at noon, 30° 30′ N.; observed altitude $\underline{\odot}$, 35° 6′; height of eye, 15 feet; time by chronometer, Aug. 5 d. 8 h. 39 m. 22 s., which was fast 29 m. 32.4 s. on Greenwich mean noon, July 8, and on July 20 was fast 30 m. 0. s. for mean noon at Greenwich; course (true) till noon, W.; distance, 48 miles. Required the longitude at noon.

15. 1895, Nov. 12, A.M., at sea, in latitude 7° 10′ N.; four observed altitudes of the $\underline{\odot}$ were taken at the times (by watch) standing opposite:

		h.	m.	s.
Obs. alt. $\underline{\odot}$:	21° 8′ 40″	2	55	48
	11′ 50″		56	0
	14′ 50″		56	13
	17′ 30″		56	26.5

Index correction, + 31″; height of eye, 18 feet; correction of watch by chronometer, − 5 h. 12 m. 2.1 s. Required the longitude.

16. 1895, Nov. 13, A.M., at sea, in latitude 9° 30′ N.; five observed altitudes of the ☉ were taken at the times (by watch) standing opposite:

		h.	m.	s.
Obs. alt. ☉:	18° 58′ 40″	2	59	2
	19° 1′ 20″			13
	3′ 30″			28
	7′ 30″			45
	11′ 0″			57.5

Index correction, + 32″; height of eye, 18 feet; correction of watch by chronometer, − 5 h. 11 m. 58.4 s. Required the longitude.

17. 1895, Nov. 17, A.M., at sea, in latitude 15° 35′ N.; five observed altitudes of the ☉ were taken at the times (by watch) standing opposite:

		h.	m.	s.
Obs. alt. ☉:	23° 56′ 0″	4	12	31
	24° 0′ 0″			46
	4′ 0″		13	2.5
	6′ 10″			14
	10′ 0″			28.5

Index correction, + 31″; height of eye, 18 feet; correction of watch by chronometer, − 5 h. 11 m. 43.6 s. Required the longitude.

18. 1895, Nov. 18, A.M., at sea, in latitude 16° 25′ N.; five observed altitudes of the ☉ were taken at the times (by watch) standing opposite:

		h.	m.	s.
Obs. alt. ☉:	18° 13′ 30″	3	52	42
	16′ 10″			53.5
	19′ 20″		53	6.5
	22′ 30″			23
	25′ 30″			38

Index correction, + 32″; height of eye, 18 feet; correction of watch by chronometer, − 5 h. 11 m. 39.9 s. Required the longitude.

19. 1895, Dec. 4, A.M., at sea, in latitude 36° 10′ N.; five observed altitudes of the ☉ were taken at the times (by watch) standing opposite:

		h.	m.	s.
Obs. alt. ☉:	13° 0′ 30″	6	27	14
	3′ 10″			29.5
	5′ 40″			49
	8′ 50″		28	5
	12′ 0″			23

Index correction, + 32″; height of eye, 18 feet; correction of watch by chronometer, — 5 h. 10 m. 47.1 s. Required the longitude.

20. 1895, Dec 4, P.M., at sea, in latitude 36° 36′ N.; four observed altitudes of the ☉ were taken at the times (by watch) standing opposite:

		h.	m.	s.
Obs. alt. ☉:	16° 31′ 10″	1	7	26.5
	30′ 0″			35.5
	28′ 30″			46
	27′ 20″			56.5

Index correction, + 30″; height of eye, 18 feet; correction of watch by chronometer, — 5 h. 10 m. 46.1 s. Required the longitude.

SECTION XXXI

DEVIATION BY TIME AZIMUTHS

The **Azimuth** of a heavenly body is the arc of the horizon included between the north point and the south point and the vertical circle passing through the body. It is named by prefixing the letter N. or S., which indicates the point from which it is reckoned, and affixing the letter E. or W., which indicates upon which side of the meridian of the observer the heavenly body is situated.

The *True Azimuth* is reckoned from the true north or south point.

The *Compass Azimuth* is reckoned from the magnetic north or south point.

When a heavenly body is on the horizon the arc of the horizon between the body and the east or west point is called the *Amplitude* of the body.

To find the compass azimuth of a heavenly body:

The compass azimuth is observed directly by means of the **Azimuth Compass.**

This instrument differs from the ordinary compass chiefly in having two sight vanes hinged to opposite points of the bowl.

The standard compass is commonly also an azimuth compass.

To observe the compass azimuth, the observer turns the compass so that by looking through the narrow slit in one of the sight vanes he can see the object, or the image of it, reflected by a small mirror, bisected by the vertical wire of the other vane. A spring is then touched which holds the card in position until the observer reads off the division of the card, apparently in the prolongation of the wire of the vane.

To find the true azimuth of a heavenly body:

First Method; by computation. The necessary data for time azimuths are: the declination (d), the hour angle (t), and the latitude (L). Let a be the true azimuth.

Formulas (A) and (C) of *Spherical Trigonometry*, Sect. LXXI, p. 193, apply.

$$\tan m = \cot d \cos t.$$

$$\tan a = \sec(L + m) \sin m \tan t.$$

NOTE. In these formulas m represents the polar distance of the foot of the perpendicular let fall from the body upon the meridian.

Second Method; by inspection. Time azimuth tables have been prepared which give the true azimuth of the sun (and

other heavenly bodies whose declinations lie between 23° N. and 23° S.) corresponding to the given latitude, declination, and apparent local time. *This is the best practical method of finding true azimuths.*

To find the total error of the compass:

The total error of the compass is equal to the difference between the true azimuth and the compass azimuth of a heavenly body when they are both reckoned from the points of like name toward the east or toward the west; it is equal to the sum of the true azimuth and the compass azimuth when one is reckoned toward the east and the other toward the west.

To name the total error of the compass, let the observer look from the centre of the compass toward the two azimuths; then the error is east or west according as the true azimuth falls to the right or left of the compass azimuth.

To find the deviation:

If the total error of the compass and the variation (given by the chart) is of the same name (E. or W.), their difference is the deviation; if of different names, their sum is the deviation.

When the error exceeds the variation, the deviation is of the same name as the error; but when the error is less than the variation, the deviation is of contrary name to the error.

The deviation is thus determined for the point of the compass toward which the ship's head lies at the moment of observation. By swinging the ship and repeating the observations the deviation may be found for as many points of the compass as may be desired. Simple interpolation may be employed in finding the deviation for a point situated between two points for which the deviations are known.

ANSWERS

PLANE TRIGONOMETRY

Exercise I. Page 2

1. $\frac{1}{3}\pi$; $\frac{1}{4}\pi$; $\frac{5}{6}\pi$; $\frac{13}{12}\pi$; $\frac{1}{16}\pi$; $\frac{11}{16}\pi$; $\frac{5}{24}\pi$.
2. 120°; 135°; 112° 30′; 168° 45′; 84°.
3. 0.017453; 0.0002909.
4. 206,265″.
5. $\frac{3}{4}\pi$; $\frac{5}{6}\pi$.
6. 11° 27′ 33″.
7. 14° 27′ 28″.
8. 69.166 miles.
9. 57 feet 3.55 inches.
10. 3 hours 49 minutes 11 seconds.
11. 9 feet 2 inches.
12. $\frac{7}{165}$ seconds.

Exercise II. Page 5

1. $\sin B = \frac{b}{c}$; $\cos B = \frac{a}{c}$; $\tan B = \frac{b}{a}$; $\cot B = \frac{a}{b}$; $\sec B = \frac{c}{a}$; $\csc B = \frac{c}{b}$.

3. (i) $\sin = \frac{3}{5}$, $\cos = \frac{4}{5}$, $\tan = \frac{3}{4}$, $\cot = \frac{4}{3}$, $\sec = \frac{5}{4}$, $\csc = \frac{5}{3}$;
   (ii) $\sin = \frac{5}{13}$, $\cos = \frac{12}{13}$, $\tan = \frac{5}{12}$, $\cot = \frac{12}{5}$, $\sec = \frac{13}{12}$, $\csc = \frac{13}{5}$;
   (iii) $\sin = \frac{8}{17}$, $\cos = \frac{15}{17}$, $\tan = \frac{8}{15}$, $\cot = \frac{15}{8}$, $\sec = \frac{17}{15}$, $\csc = \frac{17}{8}$;
   (iv) $\sin = \frac{9}{41}$, $\cos = \frac{40}{41}$, $\tan = \frac{9}{40}$, $\cot = \frac{40}{9}$, $\sec = \frac{41}{40}$, $\csc = \frac{41}{9}$;
   (v) $\sin = \frac{39}{89}$, $\cos = \frac{80}{89}$, $\tan = \frac{39}{80}$, $\cot = \frac{80}{39}$, $\sec = \frac{89}{80}$, $\csc = \frac{89}{39}$;
   (vi) $\sin = \frac{119}{169}$, $\cos = \frac{120}{169}$, $\tan = \frac{119}{120}$, $\cot = \frac{120}{119}$,
   $\sec = \frac{169}{120}$, $\csc = \frac{169}{119}$.

4. The required condition is that $a^2 + b^2 = c^2$. It is.

5. (i) $\sin = \frac{2mn}{m^2+n^2}$, $\cos = \frac{m^2-n^2}{m^2+n^2}$, $\tan = \frac{2mn}{m^2-n^2}$,
   $\cot = \frac{m^2-n^2}{2mn}$, $\sec = \frac{m^2+n^2}{m^2-n^2}$, $\csc = \frac{m^2+n^2}{2mn}$;
   (ii) $\sin = \frac{2xy}{x^2+y^2}$, $\cos = \frac{x^2-y^2}{x^2+y^2}$, $\tan = \frac{2xy}{x^2-y^2}$,
   $\cot = \frac{x^2-y^2}{2xy}$, $\sec = \frac{x^2+y^2}{x^2-y^2}$, $\csc = \frac{x^2+y^2}{2xy}$;

(iii) $\sin = \frac{q}{s}$, $\cos = \frac{q}{p}$, $\tan = \frac{p}{s}$, $\cot = \frac{s}{p}$, $\sec = \frac{p}{q}$, $\csc = \frac{s}{q}$;

(iv) $\sin = \frac{ms}{qr}$, $\cos = \frac{mpv}{nqr}$, $\tan = \frac{ns}{pv}$,

$\cot = \frac{pv}{ns}$, $\sec = \frac{nqr}{mpv}$, $\csc = \frac{qr}{ms}$.

7. In (iii) $p^2q^2 + q^2s^2 = p^2s^2$; in (iv) $m^2n^2s^2 + m^2p^2v^2 = n^2q^2r^2$.

8. $\sin A = \frac{24}{145} = \cos B$; $\cos A = \frac{143}{145} = \sin B$; $\tan A = \frac{24}{143} = \cot B$;
$\cot A = \frac{143}{24} = \tan B$; $\sec A = \frac{145}{143} = \csc B$; $\csc A = \frac{145}{24} = \sec B$.

9. $\sin A = \frac{264}{265} = \cos B$; $\cos A = \frac{23}{265} = \sin B$; $\tan A = \frac{264}{23} = \cot B$;
$\cot A = \frac{23}{264} = \tan B$; $\sec A = \frac{265}{23} = \csc B$; $\csc A = \frac{265}{264} = \sec B$.

10. $\sin A = \frac{168}{193} = \cos B$; $\cos A = \frac{95}{193} = \sin B$; $\tan A = \frac{168}{95} = \cot B$;
$\cot A = \frac{95}{168} = \tan B$; $\sec A = \frac{193}{95} = \csc B$; $\csc A = \frac{193}{168} = \sec B$.

11. $\sin A = \dfrac{\sqrt{p^2+q^2}}{p+q} = \cos B$; $\quad \cos A = \dfrac{\sqrt{2\,pq}}{p+q} = \sin B$;

$\tan A = \dfrac{\sqrt{p^2+q^2}}{\sqrt{2\,pq}} = \cot B$; $\quad \cot A = \dfrac{\sqrt{2\,pq}}{\sqrt{p^2+q^2}} = \tan B$;

$\sec A = \dfrac{p+q}{\sqrt{2\,pq}} = \csc B$; $\quad \csc A = \dfrac{p+q}{\sqrt{p^2+q^2}} = \sec B$.

12. $\sin A = \dfrac{\sqrt{p^2+pq}}{p+q} = \cos B$; $\quad \cos A = \dfrac{\sqrt{q^2+pq}}{p+q} = \sin B$;

$\tan A = \sqrt{\dfrac{p}{q}} = \cot B$; $\quad \cot A = \sqrt{\dfrac{q}{p}} = \tan B$;

$\sec A = \dfrac{p+q}{\sqrt{q^2+pq}} = \csc B$; $\quad \csc A = \dfrac{p+q}{\sqrt{p^2+pq}} = \sec B$.

13. $\sin A = \dfrac{p-q}{p+q} = \cos B$; $\quad \cos A = \dfrac{2\sqrt{pq}}{p+q} = \sin B$;

$\tan A = \dfrac{p-q}{2\sqrt{pq}} = \cot B$; $\quad \cot A = \dfrac{2\sqrt{pq}}{p-q} = \tan B$;

$\sec A = \dfrac{p+q}{2\sqrt{pq}} = \csc B$; $\quad \csc A = \dfrac{p+q}{p-q} = \sec B$.

14. $\sin A = \frac{2}{5}\sqrt{5}$; $\cos A = \frac{1}{5}\sqrt{5}$; $\tan A = 2$; $\cot A = \frac{1}{2}$;
$\sec A = \sqrt{5}$; $\csc A = \frac{1}{2}\sqrt{5}$.

15. $\sin A = \frac{2}{3}$; $\cos A = \frac{1}{3}\sqrt{5}$; $\tan A = \frac{2}{5}\sqrt{5}$; $\cot A = \frac{1}{2}\sqrt{5}$;
$\sec A = \frac{3}{5}\sqrt{5}$; $\csc A = \frac{3}{2}$.

16. $\sin A = \frac{1}{8}(5 + \sqrt{7})$; $\cos A = \frac{1}{8}(5 - \sqrt{7})$;
$\tan A = \frac{1}{9}(16 + 5\sqrt{7})$; $\cot A = \frac{1}{9}(16 - 5\sqrt{7})$;
$\sec A = \frac{4}{9}(5 + \sqrt{7})$; $\csc A = \frac{4}{9}(5 - \sqrt{7})$.

17. $\sin A = \frac{1}{8}(\sqrt{31} + 1)$; $\cos A = \frac{1}{8}(\sqrt{31} - 1)$;
$\tan A = \frac{1}{15}(16 + \sqrt{31})$; $\cot A = \frac{1}{15}(16 - \sqrt{31})$;
$\sec A = \frac{4}{15}(\sqrt{31} + 1)$; $\csc A = \frac{4}{15}(\sqrt{31} - 1)$.

18. $a = 12.3$. **20.** $a = 9$. **22.** $c = 40$.

19. $b = 1.54$. **21.** $b = 68$. **23.** $c = 229.62$.

24. Construct a rt. $\triangle$ with legs equal to 3 and 2, respectively; then construct a similar $\triangle$ with hypotenuse equal to 6.

28. $a = 1.5$ miles; $b = 2$ miles.

30. $a = 0.342$, $b = 0.940$; $a = 1.368$, $b = 3.760$. **31.** 142.926 yards.

Exercise III. Page 9

5. Through A (Fig. 3) draw a tangent, and take AT equal to 3; the angle AOT is the required angle.

6. From O (Fig. 3) as a centre, with a radius equal to 2, describe an arc cutting at S the tangent drawn through B; the angle AOS is the required angle.

7. In Fig. 3, take OM equal to $\frac{1}{2}$, and erect $MP \perp OA$, intersecting the circumference at P; the angle POM is the required angle.

8. Since $\sin x = \cos x$, $OM = PM$ (Fig. 3), and $x = 45°$; hence, construct x equal to 45°.

9. Construct a rt. $\triangle$ with one leg equal to twice the other; the angle opposite the longer leg is the required angle.

10. Divide OA (Fig. 3) into four equal parts; at the first point of division from O erect a perpendicular meeting the circumference at some point P. Draw OP; the angle AOP is the required angle.

12. $x = 18°$. **21.** $r \sin x$. **22.** $a = mc$; $b = nc$.

Exercise IV. Page 12

1. cos 60°; sin 45°; cot 1°; tan 75°;
sec 71° 50′; sin 52° 36′; tan 7° 41′; sec 35° 14′.

2. cos 30°; sin 15°; cot 33°; tan 6°;
sec 20° 58′; sin 4° 21′; tan 0° 1′; sec 44° 59′.

3. $\frac{1}{3}\sqrt{3}$.
4. 45°.
5. 30°.
6. 30°.
7. 90°.
8. 60°.
9. 22° 30′.
10. 18°.
11. 10°.
12. $\frac{90°}{n+1}$.

Exercise VI. Page 16

1. $\cos A = \frac{5}{13}$; $\tan A = \frac{12}{5}$; $\cot A = \frac{5}{12}$; $\sec A = \frac{13}{5}$; $\csc A = \frac{13}{12}$.
2. $\cos A = 0.6$; $\tan A = 1.3333$; $\cot A = 0.75$; $\sec A = 1.6667$; $\csc A = 1.25$.
3. $\sin A = \frac{11}{61}$; $\tan A = \frac{11}{60}$; $\cot A = \frac{60}{11}$; $\sec A = \frac{61}{60}$; $\csc A = \frac{61}{11}$.
4. $\sin A = 0.96$; $\tan A = 3.4286$; $\cot A = 0.2917$; $\sec A = 3.5714$; $\csc A = 1.0417$.
5. $\sin A = 0.8$; $\cos A = 0.6$; $\cot A = 0.75$; $\sec A = 1.6667$; $\csc A = 1.25$.
6. $\sin A = \frac{1}{2}\sqrt{2}$; $\cos A = \frac{1}{2}\sqrt{2}$; $\tan A = 1$; $\sec A = \sqrt{2}$; $\csc A = \sqrt{2}$.
7. $\sin A = 0.90$; $\cos A = 0.45$; $\tan A = 2$; $\sec A = 2.22$; $\csc A = 1.11$.
8. $\sin A = \frac{1}{2}\sqrt{3}$; $\cos A = \frac{1}{2}$; $\tan A = \sqrt{3}$; $\cot A = \frac{1}{3}\sqrt{3}$; $\csc A = \frac{2}{3}\sqrt{3}$
9. $\sin A = \frac{1}{2}\sqrt{2}$; $\cos A = \frac{1}{2}\sqrt{2}$; $\tan A = 1$; $\cot A = 1$; $\sec A = \sqrt{2}$.
10. $\cos A = \sqrt{1-m^2}$; $\tan A = \frac{m}{1-m^2}\sqrt{1-m^2}$; $\cot A = \frac{1}{m}\sqrt{1-m^2}$; $\sec A = \frac{1}{\sqrt{1-m^2}}$; $\csc A = \frac{1}{m}$.
11. $\cos A = \frac{1-m^2}{1+m^2}$; $\tan A = \frac{2m}{1-m^2}$; $\cot A = \frac{1-m^2}{2m}$; $\sec A = \frac{1+m^2}{1-m^2}$; $\csc A = \frac{1+m^2}{2m}$.
12. $\sin A = \frac{m^2-n^2}{m^2+n^2}$; $\tan A = \frac{m^2-n^2}{2mn}$; $\cot A = \frac{2mn}{m^2-n^2}$; $\sec A = \frac{m^2+n^2}{2mn}$; $\csc A = \frac{m^2+n^2}{m^2-n^2}$.
13. $\sin = \frac{1}{2}\sqrt{2}$; $\cos = \frac{1}{2}\sqrt{2}$; $\cot = 1$; $\sec = \sqrt{2}$; $\csc = \sqrt{2}$.
14. $\cos = \frac{1}{2}\sqrt{3}$; $\tan = \frac{1}{3}\sqrt{3}$; $\cot = \sqrt{3}$; $\sec = \frac{2}{3}\sqrt{3}$; $\csc = 2$.
15. $\sin = \frac{1}{2}\sqrt{3}$; $\cos = \frac{1}{2}$; $\tan = \sqrt{3}$; $\cot = \frac{1}{3}\sqrt{3}$; $\sec = 2$.
16. $\sin = \frac{1}{2}\sqrt{2-\sqrt{3}}$; $\cos = \frac{1}{2}\sqrt{2+\sqrt{3}}$; $\cot = 2+\sqrt{3}$; $\sec = 2(2-\sqrt{3})\sqrt{2+\sqrt{3}}$; $\csc = 2(2+\sqrt{3})\sqrt{2-\sqrt{3}}$.

17. $\sin = \frac{1}{2}\sqrt{2-\sqrt{2}}$; $\cos = \frac{1}{2}\sqrt{2+\sqrt{2}}$; $\tan = \sqrt{2}-1$;
$\sec = (2-\sqrt{2})\sqrt{2+\sqrt{2}}$; $\csc = (2+\sqrt{2})\sqrt{2-\sqrt{2}}$.

18. $\cos = 1$; $\tan = 0$; $\cot = \infty$; $\sec = 1$; $\csc = \infty$.

19. $\cos = 0$; $\tan = \infty$; $\cot = 0$; $\sec = \infty$; $\csc = 1$.

20. $\sin = 1$; $\cos = 0$; $\cot = 0$; $\sec = \infty$; $\csc = 1$.

21. $\cos A = \sqrt{1-\sin^2 A}$; $\tan A = \dfrac{\sin A}{\sqrt{1-\sin^2 A}}$; $\cot A = \dfrac{\sqrt{1-\sin^2 A}}{\sin A}$;
$\sec A = \dfrac{1}{\sqrt{1-\sin^2 A}}$; $\csc A = \dfrac{1}{\sin A}$.

22. $\sin A = \sqrt{1-\cos^2 A}$; $\tan A = \dfrac{\sqrt{1-\cos^2 A}}{\cos A}$; $\cot A = \dfrac{\cos A}{\sqrt{1-\cos^2 A}}$;
$\sec A = \dfrac{1}{\cos A}$; $\csc A = \dfrac{1}{\sqrt{1-\cos^2 A}}$.

23. $\sin A = \dfrac{\tan A}{\sqrt{1+\tan^2 A}}$; $\cos A = \dfrac{1}{\sqrt{1+\tan^2 A}}$; $\cot A = \dfrac{1}{\tan A}$;
$\sec A = \sqrt{1+\tan^2 A}$; $\csc A = \dfrac{\sqrt{1+\tan^2 A}}{\tan A}$.

24. $\sin A = \dfrac{1}{\sqrt{1+\cot^2 A}}$; $\cos A = \dfrac{\cot A}{\sqrt{1+\cot^2 A}}$; $\tan A = \dfrac{1}{\cot A}$;
$\sec A = \dfrac{\sqrt{1+\cot^2 A}}{\cot A}$; $\csc A = \sqrt{1+\cot^2 A}$.

25. $\sin A = \frac{1}{5}\sqrt{5}$; $\cos A = \frac{2}{5}\sqrt{5}$.

26. $\sin A = \frac{1}{4}\sqrt{15}$; $\tan A = \sqrt{15}$.

27. $\sin A = \frac{9}{41}$; $\cos A = \frac{40}{41}$.

28. $\dfrac{1-3\cos^2 A+3\cos^4 A}{\cos^2 A-\cos^4 A}$.

Exercise VII. Page 18

1. $x = 45°$.
2. $x = 30°$.
3. $x = 0°$, or $60°$.
4. $x = 45°$.
5. $x = 60°$.
6. $x = 45°$.
7. $x = 45°$.
8. $x = 45°$.
9. $x = 60°$.
10. $x = 60°$.
11. $x = 30°$.
12. $x = 45°$.
13. $x = 0°$, or $60°$.
14. $x = 30°$.
15. $x = 30°$, or $45°$.
16. $x = 45°$.
17. $x = 60°$.

Exercise VIII. Page 24

1. $c = \dfrac{b}{\cos A}$. 2. $c = \dfrac{a}{\sin A}$. 3. $b = c \cos A$. 4. $c = \dfrac{a}{\sin A}$.

5. $A = 90° - B$; $a = c \cos B$; $b = c \sin B$.

6. $A = 90° - B$; $a = b \cot B$; $c = \dfrac{b}{\sin B}$.

7. $A = 90° - B$; $b = a \tan B$; $c = \dfrac{a}{\cos B}$.

8. $\cos A = \dfrac{b}{c}$; $B = 90° - A$; $a = \sqrt{(c + b)(c - b)}$.

Exercise IX. Page 28

31. $c = 7.8112$; $A = 39° 48'$; $B = 50° 12'$; $F = 15$.

32. $b = 69.997$; $A = 30' 12''$; $B = 89° 29' 48''$; $F = 21.525$.

33. $a = 1.1886$; $A = 43° 20'$; $B = 46° 40'$; $F = 0.74876$.

34. $b = 21.249$; $c = 22.372$; $B = 71° 46'$; $F = 74.372$.

35. $a = 6.6882$; $c = 13.738$; $B = 60° 52'$; $F = 40.129$.

36. $a = 63.859$; $b = 23.369$; $B = 20° 6'$; $F = 746.15$.

37. $a = 19.40$; $b = 18.778$; $A = 45° 56'$; $F = 182.15$.

38. $b = 53.719$; $c = 71.377$; $A = 41° 11'$; $F = 1262.4$.

39. $a = 12.981$; $c = 15.796$; $A = 55° 16'$; $F = 58.416$.

40. $a = 0.58046$; $b = 8.442$; $A = 3° 56'$; $F = 2.4501$.

41. $F = \frac{1}{2} c^2 \sin A \cos A$.

42. $F = \frac{1}{2} a^2 \cot A$.

43. $F = \frac{1}{2} b^2 \tan A$.

44. $F = \frac{1}{2} a \sqrt{c^2 - a^2}$.

45. $b = 11.6$; $c = 15.315$; $A = 40° 45' 48''$; $B = 49° 14' 12''$.

46. $a = 7.2$; $c = 8.7658$; $A = 55° 13' 20''$; $B = 34° 46' 40''$.

47. $a = 3.6474$; $b = 6.58$; $c = 7.5233$; $B = 61°$.

48. $a = 10.283$; $b = 19.449$; $A = 27° 52'$; $B = 62° 8'$.

49. $19° 28' 17''$ and $70° 31' 43''$.

50. 3 and 5.1961.

51. $a = c \cos \dfrac{90°}{n + 1}$; $b = c \sin \dfrac{90°}{n + 1}$.

52. $36° 52' 12''$ and $53° 7' 48''$.

53. 212.1 feet.

54. 732.22 feet.

55. 3270 feet.

56. 37.3 feet.

57. 1° 25′ 56″. **58.** 59° 44′ 35″. **59.** 95.34 feet.

60. 7.0712 miles in each direction.

61. 20.88 feet. **63.** 685.9 feet. **65.** 140 feet.

62. 56.65 feet. **64.** 136.6 feet. **66.** 84.74 feet.

Exercise X. Page 33

1. $C = 2(90° - A)$; $c = 2a\cos A$; $h = a\sin A$.

2. $A = \frac{1}{2}(180° - C)$; $c = 2a\cos A$; $h = a\sin A$.

3. $C = 2(90° - A)$; $a = \dfrac{c}{2\cos A}$; $h = a\sin A$.

4. $A = \frac{1}{2}(180° - C)$; $a = \dfrac{c}{2\cos A}$; $h = a\sin A$.

5. $C = 2(90° - A)$; $a = \dfrac{h}{\sin A}$; $c = 2a\cos A$.

6. $A = \frac{1}{2}(180° - C)$; $a = \dfrac{h}{\sin A}$; $c = 2a\cos A$.

7. $\sin A = \dfrac{h}{a}$; $C = 2(90° - A)$; $c = 2a\cos A$.

8. $\tan A = \dfrac{2h}{c}$; $C = 2(90° - A)$; $a = \dfrac{h}{\sin A}$.

9. $A = 67° 22′ 50″$; $C = 45° 14′ 20″$; $h = 13.2$.

10. $c = 0.21943$; $h = 0.27384$; $F = 0.03004$.

11. $a = 2.055$; $h = 1.6852$; $F = 1.9819$.

12. $a = 7.706$; $c = 3.6676$; $F = 13.725$.

13. $A = 79° 36′ 30″$; $C = 20° 47′$; $c = 2.4206$.

14. $A = 77° 19′ 11″$; $C = 25° 21′ 38″$; $a = 20.5$.

15. $A = 25° 27′ 47″$; $C = 129° 4′ 26″$; $a = 81.41$; $h = 35$.

16. $A = 81° 12′ 9″$; $C = 17° 35′ 42″$; $a = 17$; $c = 5.2$

17. $F = \frac{1}{4}c\sqrt{4a^2 - c^2}$.

18. $F = a^2 \sin \frac{1}{2}C \cos \frac{1}{2}C$.

19. $F = a^2 \sin A \cos A$.

20. $F = h^2 \tan \frac{1}{2}C$.

21. 28.284 feet; 4525.44 square feet.

22. 0.76536.

23. 94° 20′.

24. 2.7261.

25. 38° 56′ 33″.

26. 37.699.

Exercise XI. Page 35

1. $r = 1.618$; $h = 1.5388$; $F = 7.694$.
2. $h = 0.9848$; $p = 6.2514$; $F = 3.0782$.
3. $h = 19.754$; $c = 6.257$; $F = 1236$.
4. $r = 1.0824$; $c = 0.82842$; $F = 3.3137$.
5. $r = 2.5933$; $h = 2.4882$; $c = 1.4615$.
6. $r = 1.5994$; $h = 1.441$; $p = 9.716$.
7. 0.61803.
8. 0.64984.
9. 0.51764.
10. $b = \dfrac{c}{2\cos\dfrac{90^\circ}{n}}$.
11. 0.2238.
12. 0.310.
13. 0.82842.
14. 94.63.
15. 414.97.
16. 11.636.
17. 99.640.
18. 1.0235.
19. 0.635.

Exercise XII. Page 45

5. Two angles; one in Quadrant I, one in Quadrant II.
6. Four values; two in Quadrant I, two in Quadrant IV.
7. x may have two values in the first case, and one value in each of the other cases.
8. If $\cos x = -\frac{2}{9}$, x is between 90° and 270°; if $\cot x = 4$, x is between 0° and 90° or between 180° and 270°; if $\sec x = 80$, x is between 0° and 90° or between 270° and 360°; if $\csc x = -3$, x is between 180° and 360°.
9. In Quadrant III; in Quadrant II; in Quadrant III.
10. 40 angles; 20 positive and 20 negative.
11. +, when x is known to be in Quadrant I or IV; −, when x is known to be in Quadrant II or III.
12. $\sin x = +\frac{1}{2}\sqrt{2}$; $\tan x = -1$; $\cot x = -1$; $\sec x = -\sqrt{2}$; $\csc x = +\sqrt{2}$.
13. $\sin x = -\frac{1}{2}\sqrt{3}$; $\cos x = -\frac{1}{2}$; $\cot x = +\frac{1}{3}\sqrt{3}$; $\sec x = -2$; $\csc x = -\frac{2}{3}\sqrt{3}$.
14. $\sin x = -\frac{4}{7}\sqrt{3}$; $\cos x = \frac{1}{7}$; $\tan x = -4\sqrt{3}$; $\cot x = -\frac{1}{12}\sqrt{3}$; $\csc x = -\frac{7}{12}\sqrt{3}$.
15. $\sin x = \pm\frac{1}{10}\sqrt{10}$; $\cos x = \mp\frac{3}{10}\sqrt{10}$; $\tan x = -\frac{1}{3}$; $\sec x = \mp\frac{1}{3}\sqrt{10}$; $\csc x = \pm\sqrt{10}$.

16. The cosine, the tangent, the cotangent, and the secant are negative when the angle is obtuse.

18. $\sin 90° = 1$, $\cos 90° = 0$, $\cot 90° = 0$, $\sec 90° = \infty$, $\csc 90° = 1$;
$\sin 180° = 0$, $\tan 180° = 0$, $\cot 180° = \infty$, $\sec 180° = -1$, $\csc 180° = \infty$;
$\sin 270° = -1$, $\cos 270° = 0$, $\tan 270° = \infty$, $\sec 270° = \infty$, $\csc 270° = -1$;
$\sin 360° = 0$, $\cos 360° = 1$, $\tan 360° = 0$, $\cot 360° = \infty$, $\sec 360° = 1$.

19. $\sin 450° = 1$; $\tan 540° = 0$; $\cos 630° = 0$; $\cot 720° = \infty$; $\sin 810° = 1$; $\csc 900° = \infty$.

20. 0. **21.** 0. **22.** 0. **23.** $a^2 - b^2 + 4ab$.

Exercise XIII. Page 51

1. $-\cos 20°$

2. $\sin 8°$.

3. $-\sin 10°$.

4. $-\cot 35°$.

5. $-\tan 1°$.

6. $-\csc 20°$.

7. $\csc 23°$.

8. $-\sin 24°$.

9. $\cos 1°$.

10. $-\cot 30°$.

11. $\tan 6°$.

12. $-\csc 26°$.

13. $-\sec 1°$.

14. $\sin 16° 11'$.

15. $-\cos 15° 33'$.

16. $\cot 0° 45'$.

17. $-\cot 40° 43'$.

18. $\csc 29° 45'$.

19. $\sec 2° 25'$.

20. $\sin(-75°) = -\cos 15°$;
$\cos(-75°) = \sin 15°$;
$\tan(-75°) = -\cot 15°$;
$\cot(-75°) = -\tan 15°$.

21. $\sin(-127°) = -\cos 37°$;
$\cos(-127°) = -\sin 37°$;
$\tan(-127°) = \cot 37°$;
$\cot(-127°) = \tan 37°$.

22. $\sin(-200°) = \sin 20°$;
$\cos(-200°) = -\cos 20°$;
$\tan(-200°) = -\tan 20°$;
$\cot(-200°) = -\cot 20°$.

23. $\sin(-345°) = \sin 15°$;
$\cos(-345°) = \cos 15°$;
$\tan(-345°) = \tan 15°$;
$\cot(-345°) = \cot 15°$.

24. $\sin(-52° 37') = -\cos 37° 23'$;
$\cos(-52° 37') = \sin 37° 23'$;
$\tan(-52° 37') = -\cot 37° 23'$;
$\cot(-52° 37') = -\tan 37° 23'$.

25. $\sin(-196° 54') = -\sin 16° 54'$;
$\cos(-196° 54') = -\cos 16° 54'$;
$\tan(-196° 54') = -\tan 16° 54'$;
$\cot(-196° 54') = -\cot 16° 54'$.

26. $\sin 120° = +\frac{1}{2}\sqrt{3}$; $\cos 120° = -\frac{1}{2}$; $\tan 120° = -\sqrt{3}$; $\cot 120° = -\frac{1}{3}\sqrt{3}$.

27. $\sin 135° = +\frac{1}{2}\sqrt{2}$; $\cos 135° = -\frac{1}{2}\sqrt{2}$; $\tan 135° = -1$; $\cot 135° = -1$.

28. $\sin 150° = \frac{1}{2}$; $\cos 150° = -\frac{1}{2}\sqrt{3}$; $\tan 150° = -\frac{1}{3}\sqrt{3}$; $\cot 150° = -\sqrt{3}$.

29. $\sin 210° = -\frac{1}{2}$; $\cos 210° = -\frac{1}{2}\sqrt{3}$; $\tan 210° = +\frac{1}{3}\sqrt{3}$; $\cot 210° = +\sqrt{3}$.

30. $\sin 225° = -\frac{1}{2}\sqrt{2}$; $\cos 225° = -\frac{1}{2}\sqrt{2}$; $\tan 225° = 1$; $\cot 225° = 1$.

31. $\sin 240° = -\frac{1}{2}\sqrt{3}$; $\cos 240° = -\frac{1}{2}$; $\tan 240° = +\sqrt{3}$; $\cot 240° = +\frac{1}{3}\sqrt{3}$.

32. $\sin 300° = -\frac{1}{2}\sqrt{3}$; $\cos 300° = \frac{1}{2}$; $\tan 300° = -\sqrt{3}$; $\cot 300° = -\frac{1}{3}\sqrt{3}$.

33. $\sin(-30°) = -\frac{1}{2}$; $\cos(-30°) = +\frac{1}{2}\sqrt{3}$; $\tan(-30°) = -\frac{1}{3}\sqrt{3}$; $\cot(-30°) = -\sqrt{3}$.

34. $\sin(-225°) = +\frac{1}{2}\sqrt{2}$; $\cos(-225°) = -\frac{1}{2}\sqrt{2}$; $\tan(-225°) = -1$; $\cot(-225°) = -1$.

35. $\cos x = -\frac{1}{2}\sqrt{2}$; $\tan x = 1$; $\cot x = 1$; $x = 225°$.

36. $\sin x = \frac{1}{2}$; $\cos x = -\frac{1}{2}\sqrt{3}$; $\tan x = -\frac{1}{3}\sqrt{3}$; $x = 150°$.

37. $\sin 3540° = -\frac{1}{2}\sqrt{3}$; $\cos 3540° = \frac{1}{2}$; $\tan 3540° = -\sqrt{3}$; $\cot 3540° = -\frac{1}{3}\sqrt{3}$.

38. 210° and 330°; 120° and 300°.

39. 135°, 225°, and $-225°$; 150° and $-30°$.

40. 30°, 150°, 390°, and 510°.

41. $\sin 168°$; $\cos 334°$; $\tan 225°$; $\cot 252°$; $\sin 349°$; $\cos 240°$; $\tan 64°$; $\cot 177°$.

42. 0.8480. **43.** -1.9522. **44.** $(a - b)\sin x$.

45. $m \sin x \cos x$. **48.** 0.

46. $(a - b)\cot x - (a + b)\tan x$. **49.** $\cos x \sin y - \sin x \cos y$.

47. $a^2 + b^2 + 2ab \cos x$. **50.** $\tan x$.

51. Positive between $x = 0°$ and $x = 135°$, and between $x = 315°$ and $x = 360°$; negative between $x = 135°$ and $x = 315°$.

52. Positive between $x = 45°$ and $x = 225°$; negative between $x = 0°$ and $x = 45°$, and between $x = 225°$ and $x = 360°$.

53. $\sin(x - 90°) = -\cos x$; $\cos(x - 90°) = \sin x$; $\tan(x - 90°) = -\cot x$; $\cot(x - 90°) = -\tan x$.

54. $\sin(x - 180°) = -\sin x$; $\cos(x - 180°) = -\cos x$; $\tan(x - 180°) = \tan x$; $\cot(x - 180°) = \cot x$.

Exercise XIV. Page 60

1. $\sin(x+y) = \frac{56}{65}$; $\cos(x+y) = \frac{33}{65}$.

2. $\sin(90^\circ - y) = \cos y$; $\cos(90^\circ - y) = \sin y$.

3. $\sin(90^\circ + y) = \cos y$; $\cos(90^\circ + y) = -\sin y$;
   $\tan(90^\circ + y) = -\cot y$; $\cot(90^\circ + y) = -\tan y$.

4. $\sin(180^\circ - y) = \sin y$; $\cos(180^\circ - y) = -\cos y$;
   $\tan(180^\circ - y) = -\tan y$; $\cot(180^\circ - y) = -\cot y$.

5. $\sin(180^\circ + y) = -\sin y$; $\cos(180^\circ + y) = -\cos y$;
   $\tan(180^\circ + y) = \tan y$; $\cot(180^\circ + y) = \cot y$.

6. $\sin(270^\circ - y) = -\cos y$; $\cos(270^\circ - y) = -\sin y$;
   $\tan(270^\circ - y) = \cot y$; $\cot(270^\circ - y) = \tan y$.

7. $\sin(270^\circ + y) = -\cos y$; $\cos(270^\circ + y) = \sin y$;
   $\tan(270^\circ + y) = -\cot y$; $\cot(270^\circ + y) = -\tan y$.

8. $\sin(360^\circ - y) = -\sin y$; $\cos(360^\circ - y) = \cos y$;
   $\tan(360^\circ - y) = -\tan y$; $\cot(360^\circ - y) = -\cot y$.

9. $\sin(360^\circ + y) = \sin y$; $\cos(360^\circ + y) = \cos y$;
   $\tan(360^\circ + y) = \tan y$; $\cot(360^\circ + y) = \cot y$.

10. $\sin(x - 90^\circ) = -\cos x$; $\cos(x - 90^\circ) = \sin x$;
    $\tan(x - 90^\circ) = -\cot x$; $\cot(x - 90^\circ) = -\tan x$.

11. $\sin(x - 180^\circ) = -\sin x$; $\cos(x - 180^\circ) = -\cos x$;
    $\tan(x - 180^\circ) = \tan x$; $\cot(x - 180^\circ) = \cot x$.

12. $\sin(x - 270^\circ) = \cos x$; $\cos(x - 270^\circ) = -\sin x$;
    $\tan(x - 270^\circ) = -\cot x$; $\cot(x - 270^\circ) = -\tan x$.

13. $\sin(-y) = -\sin y$; $\cos(-y) = \cos y$;
    $\tan(-y) = -\tan y$; $\cot(-y) = -\cot y$.

14. $\sin(45^\circ - y) = \frac{1}{2}\sqrt{2}(\cos y - \sin y)$; $\cos(45^\circ - y) = \frac{1}{2}\sqrt{2}(\cos y + \sin y)$;
    $\tan(45^\circ - y) = \dfrac{1 - \tan y}{1 + \tan y}$; $\cot(45^\circ - y) = \dfrac{\cot y + 1}{\cot y - 1}$.

15. $\sin(45^\circ + y) = \frac{1}{2}\sqrt{2}(\cos y + \sin y)$; $\cos(45^\circ + y) = \frac{1}{2}\sqrt{2}(\cos y - \sin y)$;
    $\tan(45^\circ + y) = \dfrac{1 + \tan y}{1 - \tan y}$; $\cot(45^\circ + y) = \dfrac{\cot y - 1}{\cot y + 1}$.

16. $\sin(30^\circ + y) = \frac{1}{2}(\cos y + \sqrt{3}\sin y)$; $\cos(30^\circ + y) = \frac{1}{2}(\sqrt{3}\cos y - \sin y)$;
    $\tan(30^\circ + y) = \dfrac{\frac{1}{3}\sqrt{3} + \tan y}{1 - \frac{1}{3}\sqrt{3}\tan y}$; $\cot(30^\circ + y) = \dfrac{\sqrt{3}\cot y - 1}{\cot y + \sqrt{3}}$.

17. $\sin(60°-y)=\frac{1}{2}(\sqrt{3}\cos y-\sin y)$; $\cos(60°-y)=\frac{1}{2}(\cos y+\sqrt{3}\sin y)$;
$\tan(60°-y)=\dfrac{\sqrt{3}-\tan y}{1+\sqrt{3}\tan y}$; $\cot(60°-y)=\dfrac{\frac{1}{3}\sqrt{3}\cot y+1}{\cot y-\frac{1}{3}\sqrt{3}}$.

18. $3\sin x-4\sin^3 x$. **19.** $4\cos^3 x-3\cos x$. **20.** 0. **21.** $\frac{1}{2}\sqrt{3}$.

22. $\sin\frac{1}{2}x=\sqrt{\dfrac{1-0.4\sqrt{6}}{2}}=0.10051$; $\cos\frac{1}{2}x=\sqrt{\dfrac{1+0.4\sqrt{6}}{2}}=0.99493$.

23. $\cos 2x=-\frac{1}{2}$, $\tan 2x=-\sqrt{3}$.

24. $\sin 22\frac{1}{2}°=\frac{1}{2}\sqrt{2-\sqrt{2}}=0.3827$; $\cos 22\frac{1}{2}°=\frac{1}{2}\sqrt{2+\sqrt{2}}=0.9239$;
$\tan 22\frac{1}{2}°=\sqrt{2}-1=0.4142$; $\cot 22\frac{1}{2}°=\sqrt{2}+1=2.4142$.

25. $\sin 15°=\frac{1}{2}\sqrt{2-\sqrt{3}}=0.2588$; $\cos 15°=\frac{1}{2}\sqrt{2+\sqrt{3}}=0.9659$;
$\tan 15°=2-\sqrt{3}=0.2679$; $\cot 15°=2+\sqrt{3}=3.7321$.

34. $\sin A+\sin B+\sin C=\sin A+\sin B+\sin[180°-(A+B)]$
$=\sin A+\sin B+\sin(A+B)$

By [20] and [12],
$=2\sin\frac{1}{2}(A+B)\ \cos\frac{1}{2}(A-B)+2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A+B)$
$=2\sin\frac{1}{2}(A+B)\,[\cos\frac{1}{2}(A-B)+\cos\frac{1}{2}(A+B)]$

By [22],
$=2\sin\frac{1}{2}(A+B)\,(2\cos\frac{1}{2}A\cos\frac{1}{2}B)$
$=4\sin\frac{1}{2}(A+B)\cos\frac{1}{2}A\cos\frac{1}{2}B$.

But $\cos\frac{1}{2}C=\cos[90°-\frac{1}{2}(A+B)]=\sin\frac{1}{2}(A+B)$.
$\therefore\ \sin A+\sin B+\sin C=4\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$.

35. Proof similar to that for 34.

38. $\dfrac{2}{\sin 2x}$.

39. $2\cot 2x$.

40. $\dfrac{\cos(x-y)}{\sin x\cos y}$.

41. $\dfrac{\cos(x+y)}{\sin x\cos y}$.

42. $\tan^2 x$.

43. $\dfrac{\cos(x-y)}{\cos x\cos y}$.

44. $\dfrac{\cos(x+y)}{\cos x\cos y}$.

45. $\dfrac{\cos(x-y)}{\sin x\sin y}$.

46. $\dfrac{\cos(x+y)}{\sin x\sin y}$.

47. $\tan x\tan y$.

Exercise XV. Page 63

1. $\sin^{-1}\frac{1}{2}\sqrt{3}=60°+2n\pi$ or $120°+2n\pi$;
$\tan^{-1}\frac{1}{3}\sqrt{3}=30°+2n\pi$ or $210°+2n\pi$;
$\operatorname{vers}^{-1}\frac{1}{2}\ \ =60°+2n\pi$ or $300°+2n\pi$;

$\cos^{-1}(-\frac{1}{2}\sqrt{2}) = 135° + 2n\pi$ or $225° + 2n\pi$;
$\csc^{-1}\sqrt{2} = 45° + 2n\pi$ or $135° + 2n\pi$;
$\tan^{-1}\infty = 90° + 2n\pi$ or $270° + 2n\pi$;
$\sec^{-1}2 = 60° + 2n\pi$ or $300° + 2n\pi$;
$\cos^{-1}(-\frac{1}{2}\sqrt{3}) = 150° + 2n\pi$ or $210° + 2n\pi$.

4. $\frac{1}{4}\sqrt{2}$. **10.** $\pm\frac{5}{13}$. **12.** $\pm\frac{1}{2}\sqrt{2}$.

8. 0°, 90°, 180°. **11.** $\pm\frac{7}{24}$. **13.** $x = 0$ or $\pm\frac{1}{2}\sqrt{3}$.

Exercise XVI. Page 67

1. If, for instance, $C = 90°$, [25] becomes $\frac{a}{c} = \sin A$.

3. $a^2 = b^2 + c^2$; $a^2 = b^2 + c^2 - 2bc$; $a^2 = b^2 + c^2 + 2bc$; a right triangle; a straight line; a straight line.

4. $b = a\cos C + c\cos A$; $a = b\cos C + c\cos B$; $c = b\cos A$.

6. 90°.

7. (i) $\frac{a-b}{a+b} = \tan(A - 45°)$; a right triangle.

(ii) $a + b = (a - b)(2 + \sqrt{3})$; an isosceles triangle with the angles 30°, 30°, 120°.

Exercise XVII. Page 69

9. 300 yards.

10. $AB = 59.564$ miles; $AC = 54.285$ miles.

11. 4.6064 miles; 4.4494 miles; 3.7733 miles.

12. 4.1501 and 8.67.

13. 6.1433 miles and 8.7918 miles.

14. 8 and 5.4723.

15. $a = 5$; $c = 9.6593$.

16. $a = 7$; $b = 8.573$.

17. Sides, 600 feet and 1039.2 feet; altitude, 519.6 feet.

18. 855 : 1607.

19. 5.438 and 6.857.

20. 15.588.

Exercise XVIII. Page 74

1. Two; one; no solution; one; two; no solution; one.

11. 420.

12. 124.617.

Exercise XIX. Page 78

11. 6.
12. 10.392.
14. 8.9212.
15. 25.
16. 3800 yards.
17. 729.67 yards.
18. 10.266 miles.
19. 5.0032 and 2.3385.
20. 26° 0′ 10″ and 14° 5′ 50″.
21. 430.85 yards.

Exercise XX. Page 83

11. $A = 36° 52′ 12″$; $B = 53° 7′ 48″$; $C = 90°$.
12. $A = B = 33° 33′ 27″$; $C = 112° 53′ 6″$.
13. $A = B = C = 60°$.
14. $A = 28° 57′ 18″$; $B = 46° 34′ 6″$; $C = 104° 28′ 36″$.
15. $A = 45°$; $B = 120°$; $C = 15°$.
16. $A = 45°$; $B = 60°$; $C = 75°$.
17. 4° 23′ 2″ W. of N., or W. of S.
18. 60°.
20. 0.88877.
21. 54.516 miles.
22. 84° 14′ 34″.
23. 54° 48′ 54″.
24. 105°; 15°; 60°.
25. 12.434 inches.

Exercise XXI. Page 87

1. 4,333,600.
2. 365.68.
3. 13,260.
4. 8160.
5. 240.
6. 26,208.
7. 15,540.
8. 29,450 or 6982.8.
9. 17.3206.
10. 10.392.
11. 0.19975.
12. $F = ab \sin A$.
13. $F = \frac{1}{4}(a^2 - b^2)\tan A$.
14. 2,421,000.
15. 30°; 30°; 120°.

Exercise XXII. Page 88

1. 21.166 miles; 24.966 miles.
2. 6.3399 miles.
3. 119.29 feet.
4. 30°.
5. 20 feet.
6. 2.6247 or 21.4587.

Exercise XXIII. Page 90

1. 106.70 feet; 142.86 feet.
2. 1023.9 feet.
3. 37° 34′ 5″.
4. 238,410 miles.
5. 861,860 miles.
6. 2922.4 miles.
7. 60°.
8. 3.2068.

9. 6.6031.
10. 199.56 feet.
11. 43.107 feet.
12. 45 feet.
13. 26° 34′.
14. 78.367 feet.
15. 75 feet.
16. 1.4446 miles.
17. 7912.8 miles.
18. 56.649 feet.
19. 69.282 feet.
20. 260.21 feet; 3690.3 feet.
21. 1.3438 miles.
22. 235.81 yards.
26. 8.0076 inches.
29. 460.46 feet.
30. 88.936 feet.
31. 13.657 miles per hour.
33. 56.564 feet.
34. 51.595 feet.
35. 101.892 feet.
37. N. 76° 56′ E.; 13.938 miles per hour.
38. 442.11 yards.
39. 255.78 feet.
40. 3121.1 feet; 3633.5 feet.
41. 529.49 feet.
42. 41.411 feet.
43. 234.51 feet.
44. 25.433 miles.
45. 294.69 feet.
46. 12,492.6 feet.
47. 6.3397 miles.
48. 210.44 feet.
50. 757.50 feet.
51. 520.01 yards.
52. 1366.4 feet.
53. 658.36 pounds; 22° 23′ 47″ with first force.
54. 88.326 pounds; 45° 37′ 16″ with known force.
57. 536.28; 500.16.
58. 345.48 feet.
59. 345.46 yards.
60. 61.23 feet.
62. 307.77 yards.
63. 19.8; 35.7; 44.5
64. 45°, 135°, 225°, or 315°.

65. $\cos A = \dfrac{-m \pm \sqrt{m^2 + 4(n+1)}}{2}$.

66. $\sin A = \sqrt{\dfrac{m^2 - n^2}{1 - n^2}}$; $\cos B = \dfrac{n}{m}\sqrt{\dfrac{1 - m^2}{1 - n^2}}$.

67. 60°, 120°, 240°, or 300°.
68. 0°, 60°, 180°, or 300°.
69. 0°, 30°, 150°, 180°, 210°, 330°.
71. $r = \dfrac{a}{2}\cot\dfrac{180°}{n}$; $R = \dfrac{a}{2}\csc\dfrac{180°}{n}$.
72. $F = \frac{1}{2}bc\sin A$.
73. $F = \frac{1}{2}c^2 \sin A \sin B \csc(A+B)$.
74. $F = \sqrt{s(s-a)(s-b)(s-c)}$.
76. 199 acres 8 square chains.
77. 210 acres 9.1 square chains.
78. 12 acres 9.78 square chains.
79. 3 acres 0.392 square chains.
80. 12 acres 3.45 square chains.
81. 4 acres 6.634 square chains.
82. 14 acres 5.54 square chains.
83. 61 acres 4.97 square chains.
84. 4 acres 6.633 square chains.
85. 13.93 chains; 23.21 chains; 32.50 chains.
86. 9 acres 0.055 square chains.
88. 876.34.
89. 1229.5.
91. 1075.3.
92. 2660.4.
93. 16,281.
94. 435.76 square feet.
95. 49,088 square feet.
96. 749.95 square feet.
97. 422.38 square feet.

98. 1834.95 square feet.
99. 26.88.
102. 6.
107. 6.
108. 6086.4 feet.
109. 5° 25′ 6″ S.; 457.49 miles.
110. 460.79 miles; 383.13 miles.
111. 228.98 miles; 11° 39′ 6″ S.
112. S. 56° 7′ 32″ E.; 202.58 miles.
113. N. 17° 25′ 22″ W.; 37° 46′ 13″ N.
114. 244.35 miles; S. 56° 10′ 49″ E.
115. 359.87 miles.
117. Long. 68° 54′ 39″ W.
118. 103.57 miles.
119. 33° 18′ 22″ N.; 36° 23′ 53″ W.
120. N. 28° 47′ 26″ E.; 1292.8 miles.
121. S. 50° 39′ 44″ W.; 250.84 miles; 20° 9′ 30″ W.
122. 38° 20′ 34″ N.; 55° 12′ 4″ W.
123. 171.14 miles; 32° 43′ 38″ W.
124. N. 36° 52′ 12″ W.; 36° 7′ 37″ W.
125. 173.18 miles; 51° 16′ 16″ S.; 34° 12′ 43″ E.
126. S. 50° 57′ 48″ E.; 47° 14′ 35″ N.; 20° 48′ 37″ W.
127. N. 53° 20′ 21″ E., 16° 6′ 57″ W.; *or* N. 53° 20′ 21″ W., 25° 53′ 3″ W.
128. N. 47° 42′ 33″ E., 19° 27′ 22″ N., 121° 50′ 34″ E.; *or* N. 47° 42′ 33″ W., 19° 27′ 22″ N., 116° 9′ 26″ E.; *or* S. 47° 42′ 33″ E., 14° 32′ 38″ N., 121° 48′ 20″ E.; *or* S. 47° 42′ 33″ W., 14° 32′ 38″ N., 116° 11′ 40″ E.
129. 359.82 miles; 359.73 miles; 359.50 miles.
130. 35° 49′ 10″ S., 22° 2′ 44″ W.; N. 61° 42′ W.; 183.16 miles.
131. 42° 15′ 29″ N., 69° 5′ 11″ W.; N. 72° 32′ 40″ E.; 44.939 miles.
132. 32° 53′ 34″ S., 13° 1′ 53″ E.; N. 72° 3′ 43″ W.; 287.16 miles.

Exercise XXIV. Page 107

(*The solutions here given are for angles less than 360°.*)

79. $\sin \frac{1}{2}x = \pm \frac{1}{5}\sqrt{5}$; $\cos \frac{1}{2}x = \pm \frac{2}{5}\sqrt{5}$.
80. $\pm\sqrt{5} - 2$.
81. $\pm \frac{1}{2}\sqrt{3}$.
82. $\pm \frac{3}{5}$ or $\pm \frac{4}{5}$.
83. $\pm 2\sqrt{2}$, $\pm \frac{1}{2\sqrt{3}}(9\sqrt{3} + 8\sqrt{2})$, or $\pm \frac{1}{2\sqrt{3}}(9\sqrt{3} - 8\sqrt{2})$.
84. $\pm \frac{1}{2}$.
85. $\frac{1}{4}(\sqrt{5} - 1)$; $\frac{1}{4}(\sqrt{5} + 1)$.
86. $(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}$.
87. $\left(\frac{1 \pm m}{2}\right)^{\frac{1}{2}}\left(1 \mp 2m\right)$.
88. $\pm \frac{1}{2}\sqrt{2}$ or $\pm \frac{1}{2}\sqrt{3}$.
89. $\frac{3}{4}$.
90. $\frac{4}{5}$ or $-\frac{3}{5}$.
91. $\pm \dfrac{a+1}{\sqrt{2a+1}}$.
92. 4.
93. $\tan(x + y)$.
94. $\dfrac{\sin x}{\sin y}$.
95. $-\tan x$.
96. $\tan^{-1}\dfrac{2x}{1-2x^2}$.
97. 2.
98. $-\tan^4 x + \cot^4 x$.
99. $x = \frac{1}{2}\pi$ or $\frac{3}{2}\pi$.
100. $x = 90°$ or $270°$.

101. $x = 21° 28'$ or $158° 32'$.

102. $x = 0°$ or $90°$.

103. $x = 30°, 150°, 199° 28'$, or $340° 32'$.

104. $x = 51° 19', 180°$, or $308° 41'$.

105. $x = 0°, 120°, 180°$, or $240°$.

106. $x = 45°, 161° 34', 225°$, or $341° 34'$.

107. $\theta = 60°, 120°, 240°$, or $300°$.

108. $\theta = 26° 34'$ or $206° 34'$.

109. $x = 30°$ or $150°$.

110. $x = 45°$ or $135°$.

111. $x = 30°, 150°$, or $270°$.

112. $x = 35° 16'', 144° 44', 215° 16'$, or $324° 44'$.

113. $x = 75° 58'$ or $255° 58'$.

114. $\theta = 60°, 180°$, or $300°$.

115. $\theta = 90°$ or $143° 8'$.

116. $x = 30°, 150°, 210°$, or $330°$.

117. $x = 30°, 150°$, or $270°$.

118. $x = 26° 34', 90°, 206° 34'$, or $270°$.

119. $x = 45°, 135°, 225°$, or $315°$.

120. $x = 45°, 135°, 225°$, or $315°$.

121. $x = 15°, 75°, 135°, 195°, 255°$, or $315°$.

122. $z = 45°, 135°, 225°$, or $315°$.

123. $x = 0°, 60°, 120°, 180°, 240°$, or $300°$.

124. $x = 27° 58', 135°, 242° 2'$, or $315°$.

125. $x = 0°, 45°, 180°$, or $225°$.

126. $x = 32° 46', 147° 14', 212° 46'$, or $327° 14'$.

127. $x = 0°, 45°, 90°, 180°, 225°$, or $270°$.

128. $x = 0°, 65° 42', 180°$, or $204° 18'$.

129. $x = 0°, 90°, 120°, 240°$, or $270°$.

130. $x = 0°, 36°, 72°, 108°, 144°, 180°, 216°, 252°, 288°$, or $324°$.

131. $x = 30°, 150°, 210°$, or $330°$.

132. $x = 60°$ or $240°$.

133. $x = 54° 44', 125° 16', 234° 44'$, or $305° 16'$.

134. $x = 105°$ or $345°$.

135. $x = \tan^{-1}\dfrac{a^2 - 1}{2a}$.

136. $x = \cos^{-1}\dfrac{-a \pm \sqrt{a^2 + 8a + 8}}{4}$.

137. $x = 135°$, $315°$, or $\frac{1}{2}\sin^{-1}(1 - a)$.

138. $x = 30°$, $60°$, $120°$, $150°$, $210°$, $240°$, $300°$, or $330°$.

139. $x = 60°$, $90°$, $120°$, $240°$, $270°$, or $300°$.

140. $x = 60°$, $90°$, $120°$, $240°$, $270°$, or $300°$.

141. $x = 120°$.

142. $x = 14° 29'$, $30°$, $150°$, or $165° 31'$.

143. $x = 60°$, $90°$, $270°$, or $300°$.

144. $x = 0°$, $20°$, $100°$, $140°$, $180°$, $220°$, $260°$, or $340°$.

145. $x = 45°$, $90°$, $135°$, $225°$, $270°$, or $315°$.

146. $x = 30°$, $60°$, $90°$, $120°$, $150°$, $210°$, $240°$, $270°$, $300°$, or $330°$.

147. $x = 0°$, $45°$, $90°$, $180°$, $225°$, or $270°$.

148. $x = 30°$, $60°$, $120°$, $150°$, $210°$, $240°$, $300°$, or $330°$.

149. $x = 30°$, $90°$, $150°$, $210°$, $270°$, or $330°$.

150. $x = 0°$, $45°$, $180°$, or $225°$.

151. $x = 45°$, $60°$, $120°$, $135°$, $225°$, $240°$, $300°$, or $315°$.

152. $x = 0°$, $45°$, $135°$, $225°$, or $315°$.

153. $x = 30°$, $90°$, $150°$, $210°$, $270°$, or $330°$.

154. $x = 8°$ or $168°$.

155. $x = 40° 12'$, $139° 48'$, $220° 12'$, or $319° 48'$.

156. $x = 30°$ or $330°$.

157. $x = 60°$, $120°$, $240°$, or $300°$.

158. $x = 30°$, $60°$, $120°$, $150°$, $210°$, $240°$, $300°$, or $330°$.

159. $x = 53° 8'$, $126° 52'$, $233° 8'$, or $306° 52'$.

160. $x = 30°$.

161. $x = 22° 37'$ or $143° 8'$.

162. $x = 0°$, $20°$, $40°$, $60°$, $80°$, $100°$, $120°$, $140°$, $160°$, $180°$, $200°$, $220°$, $240°$, $260°$, $280°$, $300°$, $320°$, or $340°$.

163. $x = 22° 30'$, $45°$, $67° 30'$, $90°$, $112° 30'$, $135°$, $157° 30'$, $202° 30'$, $225°$, $247° 30'$, $270°$, $292° 30'$, $315°$, or $337° 30'$.

164. $x = 45°$ or $225°$.

165. $x = \pm 1$ or $\pm \frac{1}{7}\sqrt{21}$.

166. $x = \frac{1}{3}\sqrt{3}$ or $-\frac{1}{2}\sqrt{3}$.

167. $x = \frac{1}{2}\sqrt{3}$.

168. $x = \frac{1}{2}$.

169. $x = 1$.

170. $x = 0$ or ± 1.

171. $x = \pm \frac{1}{2}\sqrt{2}$.

172. $x = \frac{1}{3}\sqrt{3}$.

173. $\theta = 120°$ or $240°$.

174. $x = 60°$, $120°$, $240°$, or $300°$.

175. $x = 0°$, $45°$, $135°$, $225°$, or $315°$.

176. $x = 0°, 180°, 220° 39', \text{ or } 319° 21'$.

177. $x = 0°, 60°, 120°, 180°, 240°, \text{ or } 300°$.

178. $\theta = 18°, 90°, 162°, 234°, 270°, \text{ or } 306°$.

179. $x = 0°, 30°, 90°, 150°, 180°, 210°, 270°, \text{ or } 330°$.

180. $x = 0°, 90°, 120°, 180°, 240°, \text{ or } 270°$.

181. $\theta = 0°, 74° 5', 127° 25', 180°, 232° 35', \text{ or } 285° 55'$.

182. $x = 0°, 90°, 180°, \text{ or } 270°$.

183. $x = 0°, 45°, 90°, 180°, 225°, \text{ or } 270°$.

184. $x = 0°, 45°, 120°, 135°, 180°, 225°, 240°, \text{ or } 315°$.

185. $\theta = 10° 12', 34° 48', 190° 12', \text{ or } 214° 48'$.

187. $\theta = 29° 19', 105° 41', 209° 19', \text{ or } 285° 41'$.

186. $x = 90° \text{ or } 270°$.

188. $x = 1$.

189. $x = \dfrac{b\sin\beta - a\cos\beta}{\sin(\beta-\alpha)}; \quad y = \dfrac{a\cos\alpha - b\sin\alpha}{\sin(\beta-\alpha)}$.

190. $x = \tan^{-1}\dfrac{a}{b} + \cos^{-1}\tfrac{1}{2}\sqrt{a^2+b^2}; \quad y = \tan^{-1}\dfrac{a}{b} - \cos^{-1}\tfrac{1}{2}\sqrt{a^2+b^2}$.

191. $\theta = \tan^{-1}\dfrac{a}{b}; \quad r = \sqrt{a^2+b^2}$.

192. $r = \dfrac{a}{\sin\left(\tan^{-1}\dfrac{a\cos\beta - b\sin\alpha}{a\sin\beta + b\cos\alpha} + \alpha\right)} = \dfrac{b}{\cos\left(\tan^{-1}\dfrac{a\cos\beta - b\sin\alpha}{a\sin\beta + b\cos\alpha} + \beta\right)};$

$\theta = \tan^{-1}\dfrac{a\cos\beta - b\sin\alpha}{a\sin\beta + b\cos\alpha}$.

193. $r = \sqrt{a^2+b^2+c^2}; \quad \theta = \tan^{-1}\dfrac{a}{b}; \quad \phi = \tan^{-1}\dfrac{c}{\sqrt{a^2+b^2}}$.

194. $x = 100; \quad y = 200$.

195. $x = 76° 10'; \quad y = 15° 30'$.

196. $r = 225.12, \ \theta = 24° 13'; \text{ or } r = -225.12, \ \theta = 204° 13'$.

197. $r = 151, \ \theta = 42° 28'; \text{ or } r = -151, \ \theta = 222° 28'$.

198. $r = 108, \ \phi = 120°, \ \theta = 330°; \ r = 108, \ \phi = 300°, \ \theta = 210°;$
$r = -108, \ \phi = 120°, \ \theta = 150°; \text{ or } r = -108, \ \phi = 300°, \ \theta = 30°$.

199. $x = r\,\text{vers}^{-1}\left(\dfrac{y}{r}\right) \mp \sqrt{2ry - y^2}$.

Exercise XXV. Page 121

1. $\log_{10} 6 = 0.77815; \quad \log_{10} 14 = 1.14613; \quad \log_{10} 21 = 1.32222;$
$\log_{10} 4 = 0.60206; \quad \log_{10} 12 = 1.07918; \quad \log_{10} 5 = 0.69897;$
$\log_{10} \tfrac{1}{2} = \bar{1}.69897; \quad \log_{10} \tfrac{1}{4} = \bar{1}.39794; \quad \log_{10} \tfrac{7}{9} = \bar{1}.89086;$
$\log_{10} \tfrac{21}{20} = 0.02119$.

2. $\log_2 10 = 3.3219$; $\log_2 5 = 2.3219$; $\log_3 5 = 1.4650$;
$\log_7 \frac{1}{2} = -0.3562$; $\log_5 \frac{9}{343} = -2.2620$.

3. $\log_e 2 = 0.69315$; $\log_e 3 = 1.09862$; $\log_e 5 = 1.60945$;
$\log_e 7 = 1.94593$; $\log_e 8 = 2.07946$; $\log_e 9 = 2.19724$;
$\log_e \frac{2}{3} = -0.40547$; $\log_e \frac{4}{5} = -0.22315$; $\log_e \frac{35}{27} = 0.25952$;
$\log_e \frac{7}{60} = -2.14845$.

4. $x = 1.54396$; $x = 0.83048$; $x = 0.42062$.

Exercise XXVI. Page 126

1. $\log_e 3 = 1.09861$. **2.** $\log_e 5 = 1.60944$. **3.** $\log_e 7 = 1.94591$.

4. $\log_e 10 = 2.3025850930$.

5. $\log_{10} 2 = 0.30103$; $\log_{10} e = 0.43429$; $\log_{10} 11 = 1.04139$.

Exercise XXVII. Page 128

1. $\sin 1' = 0.00029088820$; $\cos 1' = 0.99999995769$;
$\tan 1' = 0.000290888212$.

2. $\sin 2' = 0.000581776$. **3.** $\sin 1° = 0.0175$. **6.** 0° 40′ 9″.

Exercise XXVIII. Page 130

1. $\sin 6' = 0.0017453$; $\cos 6' = 0.9999985$.

2. $\sin 2° = 0.034902$; $\cos 2° = 0.999392$.

3. $\sin 3° = 0.052339$; $\cos 3° = 0.998632$.

4. $\sin 4° = 0.069760$; $\cos 4° = 0.997564$.

5. $\sin 5° = 0.087160$; $\cos 5° = 0.996193$.

Exercise XXIX. Page 135

1. The 6 sixth roots of -1 are:

$$\frac{\sqrt{3}+i}{2},\ i,\ \frac{-\sqrt{3}+i}{2},\ \frac{-\sqrt{3}-i}{2},\ -i,\ \frac{\sqrt{3}-i}{2}.$$

The 6 sixth roots of $+1$ are:

$$1,\ \frac{1+\sqrt{-3}}{2},\ \frac{-1+\sqrt{-3}}{2},\ -1,\ \frac{-1-\sqrt{-3}}{2},\ \frac{1-\sqrt{-3}}{2}.$$

2. $\frac{\sqrt{3}+i}{2}$, $\frac{-\sqrt{3}+i}{2}$, $-i$.

3. $\cos 67\frac{1}{2}° + i \sin 67\frac{1}{2}°$, $\cos 157\frac{1}{2}° + i \sin 157\frac{1}{2}°$, $\cos 247\frac{1}{2}° + i \sin 247\frac{1}{2}°$, $\cos 337\frac{1}{2}° + i \sin 337\frac{1}{2}°$.

4. $\sin 4\theta = 4 \cos^3\theta \sin\theta - 4 \cos\theta \sin^3\theta$;
$\cos 4\theta = \cos^4\theta - 6 \cos^2\theta \sin^2\theta + \sin^4\theta$.

Exercise XXX. Page 137

5. $\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots$

6. $x \cot x = 1 - \frac{x^2}{3} - \frac{x^4}{45} - \frac{2x^6}{945} - \cdots$

7. $\sin 10° = 0.173648$; $\cos 10° = 0.984808$.

8. $\tan 15° = 0.267949$.

SPHERICAL TRIGONOMETRY

Exercise XXXI. Page 142

1. 110°; 100°; 80°.

2. 140°; 90°; 55°.

5. Multiply their measures in degrees by $\frac{\pi r}{180}$.

6. $\frac{8}{9}\pi$ feet; 2π feet; $\frac{25}{9}\pi$ feet.

Exercise XXXII. Page 146

3. (i) Either a or $b = 90°$; (ii) $A = 90°$ and $B = b$;
(iii) $A = 90°$ and $B = b$; (iv) $c = 90°$ and $b = B = 90°$.

Exercise XXXIII. Page 148

2. Rule I. The cosine of any middle part is equal to the product of the cotangents of the adjacent parts.
Rule II. The cosine of any middle part is equal to the product of the sines of the opposite parts.

Exercise XXXIV. Page 153

24. $A = 175° 57' 10''$; $B = 135° 42' 50''$; $C = 135° 34' 7''$.

25. $a = 104° 53' 2''$; $b = 133° 39' 48''$; $C = 104° 41' 39''$.

26. $a = 90°$; $b = B$; b and B are otherwise indeterminate.

27. $a = 60°$; $b = 90°$; $B = 90°$.

28. The triangle is impossible.

29. $b = 130° 41' 42''$; $c = 71° 27' 43''$; $A = 112° 57' 2''$.

30. $a = 26° 3' 51''$; $A = 35°$; $B = 65° 46'$.

31. The triangle is impossible.

32. $a = 60° 16' 17''$; $b = 29° 41' 4''$; $B = 33° 16' 54''$.

33. $b = 42° 10' 17''$; $c = 106° 37' 37''$; $A = 105° 41' 39''$.

34. $a = 113° 51' 5''$; $c = 105° 37' 54''$; $B = 50° 44' 19''$.

35. $a = 124° 10' 37''$; $b = 107° 7' 22''$; $c = 80° 28' 49''$.

Exercise XXXV. Page 157

1. $\cos A = \cot a \tan \frac{1}{2} b$; $\sin \frac{1}{2} B = \csc a \sin \frac{1}{2} b$; $\cos h = \cos a \sec \frac{1}{2} b$.

2. $\sin \frac{1}{2} A = \frac{1}{2} \sec \frac{1}{2} a$.

3. $\sin \frac{1}{2} A = \sec \frac{1}{2} a \cos \frac{180°}{n}$; $\sin R = \sin \frac{1}{2} a \csc \frac{180°}{n}$;
$\sin r = \tan \frac{1}{2} a \cot \frac{180°}{n}$.

4. Tetrahedron, 70° 31′ 46″; hexahedron, 90°; octahedron, 109° 28′ 14″; dodecahedron, 116° 33′ 45″; icosahedron 138° 11′ 36″.

5. $\cot \frac{1}{2} A = \sqrt{\cos a}$.

Exercise XXXVI. Page 160

1. (i) $\sin a \sin B = \sin b$, $\sin a \sin C = \sin c$;
(ii) $\sin a = \sin b \sin A$, $\sin b \sin C = \sin c$;
(iii) $\sin a = \sin c \sin A$, $\sin b = \sin c \sin B$;
(iv) $\sin B = \sin b \sin A$, $\sin C = \sin c \sin A$;
(v) $\sin a = \sin b$, $\sin c = \sin a \sin C = \sin b \sin C$;
(vi) $\sin B = \sin A$, $\sin C = \sin c \sin A = \sin c \sin B$.

2. (i) $\cos a = \cos b \cos c$; (ii) $\cos b = \cos a \cos c$; (iii) $\cos c = \cos a \cos b$;
(iv) $\cos a = \cos b \cos c$, $\cos b = \cos a \cos c$, $\cos c = \cos a \cos b$.

3. (i) $\cos a = \cos (b - c)$; (ii) $\cos a = \cos b \cos c$; (iii) $\cos a = \cos (b + c)$.

Exercise XXXVII. Page 167

1. (i) $\tan m = \tan b \cos A$, $\cos a = \cos b \sec m \cos (c - m)$;
(ii) $\tan m = \tan c \cos B$, $\cos b = \cos c \sec m \cos (a - m)$.

Exercise XXXVIII. Page 169

1. (i) $\cot x = \tan B \csc a$, $\cos A = \cos B \csc x \sin (C - x)$;
(ii) $\cot x = \tan C \csc b$, $\cos B = \cos C \csc x \sin (A - x)$.

Exercise XLIII. Page 180

4. 2.2298 R^2.
5. 1.4956 R^2 or 0.17085 R^2.
6. 0.95484 R^2.
7. 0.024832 R^2.
8. 1.1891 R^2.
9. 0.7105 R^2.
10. 0.09301 R^2.
11. 2.8624 R^2.
12. 3.1416 R^2.
13. 5.4206 R^2.
14. 2070.1 square miles.

Exercise XLIV. Page 197

1. 148° 42′.

2. $\cos x = \cos A \cos B$.

3. Let w equal the inclination of the edge c to the plane of a and b. Then it is easily shown that $V = abc \sin l \sin w$. Now, conceive a sphere constructed having for centre the vertex of the trihedral angle whose edges are a, b, c. The spherical triangle, whose vertices are the points where a, b, c meet the surface of this sphere, has for its sides, l, m, n; and w is equal to the perpendicular arc to the side l from the opposite vertex. Let L, M, N denote the angles of this triangle. Then, by [39] and [47],

$$\begin{aligned} \sin w &= \sin m \sin N \\ &= 2 \sin m \sin \tfrac{1}{2} N \cos \tfrac{1}{2} N \\ &= \frac{2}{\sin l} \sqrt{\sin s \sin (s - l) \sin (s - m) \sin (s - n)}, \end{aligned}$$

where $s = \tfrac{1}{2}(l + m + n)$.

$$\therefore V = 2abc \sqrt{\sin s \sin (s - l) \sin (s - m) \sin (s - n)}.$$

4. (i) 9,976,500 square miles; (ii) 13,316,560 square miles.

5. Let m equal the longitude of the point where the ship crosses the equator, B her course at the equator, d the distance sailed.

Then,
$$\begin{aligned} \tan m &= \sin l \tan a, \\ \cos B &= \cos l \sin a, \\ \cot d &= \cot l \cos a. \end{aligned}$$

6. Let k equal the arc of the parallel between the places, x the difference required.

Then,
$$\begin{aligned} x &= 2 \cos l \sin^{-1} (\sin \tfrac{1}{2} d \sec l) - d. \\ x &= 90° (\sqrt{2} - 1). \end{aligned}$$

7. $\tan \frac{1}{2}(m - m') = \sqrt{\sec s \sec (s - d) \sin (s - l) \sin (s - l')}$; where $2s = l + l' + d$, and m and m' are the longitudes of the places.

9. 44 minutes past 12 o'clock. 10. 60°.

11. $\cos t = -\tan d \tan l$; time of sunrise $= 12 - \frac{t}{15}$ o'clock A.M.; time of sunset $= \frac{t}{15}$ o'clock P.M.; $\cos a = \sin d \sec l$. For longest day at Boston: time of sunrise, 4 hr. 26 min. 50 sec. A.M.; time of sunset, 7 hr. 33 min. 10 sec. P.M. Azimuth of sun at these times, 57° 25′ 15″; length of day, 15 hr. 6 min. 20 sec.; for shortest day, times of sunrise and sunset are 7 hr. 33 min. 10 sec. A.M. and 4 hr. 26 min. 50 sec. P.M.; azimuth of sun, 122° 34′ 45″; length of day, 8 hr. 53 min. 40 sec.

12. The problem is impossible when $\cot d < \tan l$; that is, for places in the frigid zones.

13. For the northern hemisphere and positive declination,
$\sin h = \sin l \sin d$, $\cot a = \cos l \tan d$; $h = 17° 14′ 35″$, $a = 73° 51′ 34″$ E.

14. The farther the place from the equator, the greater the sun's altitude at 6 A.M. in summer. At the equator it is 0°. At the north pole it is equal to the sun's declination. At a given place the sun's altitude at 6 A.M. is a maximum on the longest day of the year, and then $\sin h = \sin l \sin e$ (where $e = 23° 27′$).

15. $\cos t = \cot l \tan d$. Times of bearing due east and due west are $12 - \frac{t}{15}$ o'clock A.M. and $\frac{t}{15}$ o'clock P.M., respectively; 6 hr. 58 min. 10 sec. A.M. and 5 hr. 1 min. 50 sec. P.M.

16. When the days and nights are equal, $d = 0°$, $\cos t = 0$, $t = 90°$; that is, sun is everywhere due east at 6 A.M., and due west at 6 P.M. Since l and d must both be less than 90°, $\cos t$ cannot be negative, therefore t cannot be greater than 90°. As d increases, t decreases; that is, the times in question both approach noon.

If $l < d$, then $\cos t > 1$; therefore this case is impossible.

If $l = d$, then $\cos t = 1$, and $t = 0°$; that is, the times both coincide with noon. The explanation of this result is, that for $l = d$ the sun at noon is in the zenith, and south of the prime vertical at every other time.

If $l > d$, the diurnal circle of the sun and the prime vertical of the place meet in two points which separate farther and farther as l increases. At the pole the prime vertical is indeterminate; but near the pole, $t = 90°$, and the sun is always east at 6 A.M.

17. $\sin l = \sin d \csc h$.

18. 11° 50′ 35″.

19. The bearing of the wall, reckoned from the north point of the horizon, is given by the equation $\cot x = \cos l \tan d$; whence, for the given case, $x = 75° 12′ 28″$.

20. 55° 45′ 6″ N.

21. 63° 23′ 41″ N. or S.

22. (i) $\cos t = -\tan l \cot p$; (ii) $t = z$; (iii) the result is indeterminate.

23. $\cot a = \cos l \tan d$.

28. $\sin d = \sin e \sin v$; $\tan r = \cos e \tan v$.

25. $h = 65° 37′ 20″$.

29. $d = 32° 24′ 12″$; $r = 301° 48′ 17″$.

26. $h = 58° 25′ 8″$; $a = 152° 28′$.

30. $d = 20° 48′ 38″$.

27. $t = 45° 42′$; $l = 67° 58′ 54″$.

31. 3 hr. 59 min. $27\frac{4}{15}$ sec. P.M.

32. $\cos \frac{1}{2} a = \sqrt{\cos \frac{1}{2}(l + h + p) \cos \frac{1}{2}(l + h - p) \sec l \sec h}$.

SURVEYING

Exercise II. Page 230

2. 540°.

4. N. 51° 30′ E.

Exercise III. Page 235

2. 360°.

3. 235 ft. 3.8 in.

Exercise V. Page 263

1. 8 A. 64 P.

2. 16 A. $74\frac{2}{25}$ P.

3. 4 A. $5\frac{3}{25}$ P.

4. $115\frac{7}{20}$ P.

5. 3 A. 78 P.

6. 13 A. $6\frac{1}{10}$ P.

7. 2 A. $58\frac{1}{2}$ P.

8. 11 A. 157 P.

9. 7.51925.

10. 13.0735.

11. 4 A. 35 P.

12. 4 A. 110 P.

13. 6 A. $23\frac{17}{25}$ P.

Exercise VI. Page 272

1. 2 A. 26 P.
2. 20 A. 12 P.
3. 8 A. 54 P.
4. 3 A. 122 P.
5. 2 A. 78 P.
6. 6 A. 2 P.
7. 5 A. 42 P.
8. 2 A. 151 P.

Exercise VII. Page 274

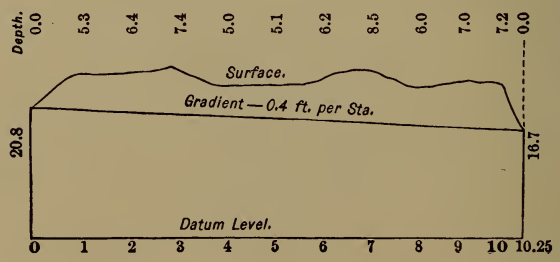
1. 2 A. $12\frac{1}{2}$ P.
2. 98 A. 92 P.

Exercise VIII. Page 277

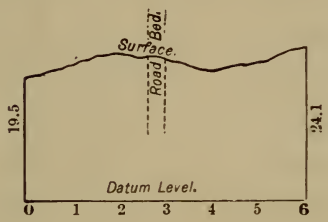
1. $AE = 3.75$ ch.
2. $AE = 3.50$ ch.; $EG = 3.42$ ch.
3. $AE = 4.55$ ch.
4. $AE = 5.50$ ch.
5. $CE = 4.456$ ch.
6. $AD = 2.275$ ch.; $BE = 1.82$ ch.
7. $AD = 4.51$ ch.; $BE = 3.61$ ch.
8. The distances on AB are 2 ch., 3 ch., and 5 ch.
9. EM (on EA) $= 2.5087$ ch.; AN (on AB) $= 6.4390$ ch.
10. Let $EG > DF$; then $AE = 12.247$ ch., $AG = 9.798$ ch., $AD = 8.660$ ch., $AF = 6.928$ ch.
11. Let $DG > EF$; then $CG = 14.862$ ch., $CD = 13.113$ ch., $CF = 11.404$ ch., $CE = 10.062$ ch.

Exercise IX. Page 295

1. 9.5.
2. Column H.G. 20.8, 20.4, 20.0, 19.6, 19.2, 18.8, 18.4, 18.0, 17.6, 17.2, 16.8, 16.7.
   Column C. 0.0, 5.3, 6.4, 7.4, 5.0, 5.1, 6.2, 8.5, 6.0, 7.0, 7.2, 0.0.



5. Third column: 26.944 opposite 0; 25.286 opposite 4.
Fifth column: 20, 19.5, 21.3, 23, 22.3, 21.431, 20.4, 21.8, 24.1.



Exercise X. Page 301

1. 9986.5 cu. yd.; 9994.9 cu. yd.
2. 5730 ft.; 2865 ft.; 1910 ft.; 1433 ft.; 1146 ft.
3. 1365 ft.; 4° 11′ 53″.

NAVIGATION

Exercise I. Page 319

1. S. 35° 56′ E.
2. N. 77° 45′ W.
3. S.E. by E.
4. S. 79° 4′ W.
5. N. 82° 47′ W.
6. S. 60° 30′ E.
7. N. 56° 22′ E.
8. S. 73° 26′ W.
9. S. 39° 15′ E.
10. N.E. by E.
11. N.N.W.
12. W.
13. S. 42° 0′ E.
14. N. 30° 0′ W.
15. N. 52° 0′ W.
16. S. 27° 0′ W.
17. S. 17° 0′ W.
18. N. 17° 56′ E.

Exercise II. Page 332

1. $L'' = 45°\ 26'$ N., $p = 405.8$ W.
2. $L'' = 2°\ 46'$ S., $p = 405.6$ E.
3. $L'' = 0°\ 3'$ S., $p = 406.0$ E.
4. $L'' = 2°\ 12'$ N., $p = 52.1$ E.
5. $L'' = 36°\ 57'$ N., $p = 113.1$ W.

6.	$C = 61° 49'$,	$p = 317.3$.
7.	$C = 25° 19'$,	$p = 151.4$.
8.	$D = 118.3$,	$p = 23.1$ W.
9.	$D = 119.6$,	$p = 57.9$ W.
10.	$L'' = 37° 28'$ N.,	$D = 58.0$.
11.	$L'' = 17° 54'$ N.,	$D = 72.2$.
12.	$C = 25° 27'$,	$p = 128.5$.
13.	$C = 22° 21'$,	$p = 308.8$.
14.	$L'' = 6° 3'$ N.,	$p = 95.4$ E.
15.	$L'' = 14° 3'$ S.,	$p = 137.5$ E.
16.	$L'' = 40° 12'$ S.,	$p = 239.7$ E.
17.	$C =$ S. $53° 22'$ W.,	$D = 360.4$.
18.	$C =$ N. $56° 6'$ E.,	$D = 154.2$.
19.	$C =$ N. $75° 5'$ E.,	$D = 147.7$.
20.	$C =$ N. $34° 20'$ W.,	$D = 93.3$.

Exercise III. Page 335

1. 505.4 E.	5. 80.1 W.	9. $172° 15'$ E.
2. 125.0 E.	6. $166° 50'$ E.	10. $12° 47'$ W.
3. 109.1 W.	7. $30° 13'$ W.	11. $179° 11'$ E.
4. 246.0 E.	8. $179° 48'$ E.	12. $21° 18'$ W.; N. $58° 40'$ E.

Exercise IV. Page 340

1.	$C =$ E.N.E.,	$D = 295.3$.
2.	$C =$ N. $61° 53'$ W.,	$D = 212.2$.
3.	$C =$ S. $64° 34'$ W.,	$D = 209.6$.
4.	$L'' = 49° 10'$ S.,	$\lambda'' = 176° 0'$ W.
5.	$L'' = 18° 52'$ N.,	$\lambda'' = 175° 12'$ E.
6.	$L'' = 0° 59'$ S.,	$\lambda'' = 27° 47'$ W.
7.	$L'' = 42° 11'$ N.,	$\lambda'' = 65° 46'$ W.
8.	$L'' = 41° 56'$ N.,	$\lambda'' = 63° 0'$ W.
9.	$L'' = 41° 36'$ N.,	$\lambda'' = 59° 12'$ W.

10. $\lambda'' = 57° 55'$ W., $D = 61.5$.
11. $\lambda'' = 61° 29'$ W., or $54° 5'$ W., $C = 87° 15'$.
12. $C =$ S. $57° 19'$ E. $D = 131.5$.
13. $C =$ S. $71° 9'$ E. $D = 510.7$.
14. $C =$ S. $53° 46'$ E., $D = 289.3$.
15. $C =$ N. $36° 25'$ W., $D = 156.6$.
16. $L'' = 45° 10'$ N., $\lambda'' = 69° 27'$ W.
17. $L'' = 35° 14'$ N., $\lambda'' = 73° 2'$ W.
18. $L'' = 42° 24'$ N., $\lambda'' = 66° 15'$ W.
19. $L'' = 39° 59'$ N., $\lambda'' = 68° 35'$ W.
20. $L'' = 53° 20'$ N., $\lambda'' = 1° 25'$ W.
21. $L'' = 30° 4'$ N., $\lambda'' = 67° 27'$ W.
22. $L'' = 33° 8'$ N., $\lambda'' = 66° 19'$ W.
23. $L'' =$ N. $64° 55'$ W., $D = 668.5$.
24. $L'' = 25° 42'$ N., $C =$ S. $36° 27'$ W.
    $D = 807.9$.

Exercise V. Page 347

1. $C =$ N. $46° 29'$ E., $D = 140.9$.
2. $C =$ S. $30° 23'$ E., $D = 98.5$.
3. $C =$ S. $28° 24'$ E., $D = 261.5$.
4. $C =$ N. $55° 13'$ E., $D = 2160$.
5. $C =$ N. $43° 39'$ W., $D = 248.8$.
6. $L'' = 47° 10'$ N., $\lambda'' = 32° 15'$ W.
7. $L'' = 59° 3'$ N., $\lambda'' = 2° 43'$ E.
8. $L'' = 47° 20'$ N., $\lambda'' = 13° 45'$ W.
9. $C =$ N. $33° 20'$ W., $\lambda'' = 61° 30'$ W., or $35° 10'$ W.
10. $\lambda'' = 23° 5'$ W., $D = 1022$.
11. $C =$ S. $51° 59'$ W., $D = 358.8$.
12. $C =$ N. $16° 2'$ E., $D = 5599$.
13. $C =$ S. $60° 29'$ E., $D = 7685$.
14. $\lambda_d = 4° 48'$, $D = 476.3$.
15. By Mercator's, $L'' = 37° 16'$ N., $\lambda'' = 22° 47'$ W.;
    Mid. Lat., $L'' = 37° 16'$ N.. $\lambda'' = 22° 43'$ W.

Exercise VI. Page 351

1. $L_d = 0° 57' S., p = 21.4 W.$
2. $L_d = 1° 37' S., p = 45.6 W.$
3. $L_d = 1° 39' S., p = 44.6 W.$
4. $L_d = 0° 45' N., p = 37.1 E.$
5. $L_d = 0° 4' S., p = 25.0 W.$
6. $L_d = 0° 15' N., p = 36.5 W.$
7. $L_d = 0° 7' S., p = 6.8 W.$
8. $L_d = 0° 11' S., p = 110.7 E.$
9. $L_d = 1° 2' S., p = 119.5 E.$

Exercise VII. Page 359

1. $L'' = 45° 18' N., \lambda'' = 19° 39' W.$
2. $L'' = 30° 49' N., \lambda'' = 17° 56' W.$
3. $L'' = 17° 24' S., \lambda'' = 1° 29' W.$
4. $L'' = 46° 51' N., \lambda'' = 48° 6' W.$
5. $L'' = 61° 34' N., \lambda'' = 149° 49' E.$
6. $L'' = 51° 29' S., \lambda'' = 176° 35' E.$
7. $L'' = 36° 31' N., \lambda'' = 2° 10' W.$

Exercise VIII. Page 374

1. Course from the Bermudas = N. 49° 32′ E.;
   Course from the Lizard = N. 88° 42′ W.;
   Distance = 2812 miles; L of V = 49° 59′ N.;
   λ of V = 6° 54′ W.

2. Course from Boston = N. 55° 24′ E.;
   Course from Cape Clear = N. 77° 42′ W.;
   Distance = 2500 miles; L of V = 52° 28′ N.;
   λ of V = 25° 2′ W.

3. Course from Honolulu = N. 32° 3′ E.;
   Course from Vancouver = S. 50° 17′ W.;
   Distance = 2228 miles; L of V = 60° 22′ N.
   λ of V = 80° 41′ W.

4. Course from Sandy Hook = N. 52° 50′ E.;
   Course from Cape Clear = N. 78° 18′ W.;
   Distance = 2698 miles; L of V = 52° 23′ N.;
   λ of V = 24° 18 W.

5. Course from Cape Frio = N. 23° 2′ E.;
Course from the Lizard = S. 34° 4′ W.;
Distance = 4806 miles; L of V = 68° 53′ N.;
λ of V = 57° 26′ E.

6. Course from Cape Good Hope = S. 85° 13′ W.;
Course from Cape Frio = S. 63° 23′ E.;
Distance = 3208 miles; L of V = 34° 37′ S.;
λ of V = 10° 3′ E.

7. Course from Perth = S. 87° 45′ E.;
Course from Grand Port = S. 64° 37′ W.;
Distance = 3158 miles; L of V = 32° 8′ S.;
λ of V = 111° 29′ E.

8. Course from A = N. 43° 57′ E.;
Course from B = S. 83° 51′ W.;
Distance = 3714 miles; L of V = 48° 19′ N.;
λ of V = 3° 39′ E.

9. By Great Circle:
Course from A = S. 49° 28′ E.;
Course from B = S. 36° 32′ W.;
Distance = 6772 miles; L of V = 54° 24′ S.;
λ of V = 158° 26′ W.

By Rhumb-line:
Course from A = N. 76° 58′ E.;
Course from B = S. 76° 58′ W.;
Distance = 7432 miles.

Exercise IX. Page 387

1. 25° 1′ 20″.
2. 15° 10′ 41″.
3. 18° 27′ 11″.
4. 30° 23′ 4″.
5. 56° 35′ 28″.
6. 60° 19′ 5″.
7. 31° 35′ 13″.
8. 25° 57′ 27″.
9. 20° 13′ 41″.
10. 35° 51′ 13″.

Exercise X. Page 389

		d.	h.	m.	s.	
1.	July	8	7	6	10	P.M.
2.	Mar.	8	0	25	30	A.M.
3.	Jan.	2	6	10	10	A.M.
4.	Jan.	1	3	0	0	A.M.
5.	Feb.	2	8	4	30	P.M.
6.	June	30	23	8	25.	
7.	Mar.	2	11	56	56.	
8.	Aug.	31	10	8	20.	
9.	Aug.	31	12	12	15.	
10.	Dec.	31	22	41	56.	

Exercise XI. Page 391

	d.	h.	m.	s.			d.	h.	m.	s.	
1.	May	4	17	35	35.	6.	Dec.	31	6	7	0 A.M.
2.	July	31	1	53	50.	7.	July	4	11	55	0 P.M.
3.	July	31	19	32	58.	8.	July	4	11	55	0 P.M.
4.	Mar.	2	6	57	0.	9.	May	20	7	43	20 A.M.
5.	Mar.	25	17	49	42.	10.	Jan.	1	3	16	40 A.M.

Exercise XII. Page 396

		m.	s.	
1.	22° 21′ 28.1″ S.;	6	32.79	to be added to app. time.
2.	17° 58′ 4.7″ N.;	6	6.95	to be added to app. time.
3.	16° 23′ 7.9″ N.;	3	28.48	to be taken from app. time.
4.	16° 15′ 35.0″ N.;	5	30.97	to be added to app. time.
5.	22° 17′ 40.7″ S.;	9	33.83	to be taken from app. time.
6.	19° 55′ 0.2″ N.;	6	14.30	to be added to app. time.
7.	14° 29′ 26.9″ S.;	16	18.84	to be taken from app. time.
8.	7° 30′ 22.7″ S.;	13	30.16	to be taken from app. time.
9.	22° 46′ 38.4″ N.;	1	25.08	to be added to mean time.
10.	16° 23′ 34.9″ S.;	14	4.88	to be taken from mean time.

Exercise XIII. Page 404

1. 6° 48′ 11″ N.
2. 5° 26′ 31″ S.
3. 0° 3′ 44″ S.
4. 45° 43′ 41″ N.
5. 47° 13′ 47″ N.
6. 43° 40′ 30″ S.
7. 15° 58′ 50″ N.
8. 0° 9′ 53″ N.
9. 4° 31′ 44″ N.
10. 49° 14′ 14″ N.
11. 5° 39′ 37″ S.
12. 6° 10′ 58″ N.
13. 29° 24′ 38″ S.
14. 0° 25′ 35″ S.
15. 50° 19′ 10″ S.
16. 7° 51′ 0″ N.
17. 6° 31′ 27″ S.

Exercise XIV. Page 405

1. 53° 37′ 33″ N.
2. 18° 15′ 19″ N.
3. 34° 9′ 18″ S.
4. 14° 36′ 16″ S.
5. 60° 34′ 36″ S.
6. 16° 14′ 49″ N.
7. 50° 32′ 48″ N.
8. 29° 34′ 14″ S.
9. 52° 35′ 28″ N.
10. 35° 41′ 47″ N.
11. 45° 35′ 1″ S.
12. 48° 28′ 5″ S.
13. 46° 6′ 32″ S.

Exercise XV. Page 413

1. 16° 54′ 0″ E.
2. 13° 53′ 0″ E.
3. 12° 55′ 45″ E.
4. 10° 54′ 30″ E.
5. 49° 24′ 45″ W.
6. 41° 28′ 15″ W.
7. 19° 58′ 30″ W.
8. 35° 26′ 0″ E. at sights.
   35° 15′ 11″ E. at noon.
9. 170° 58′ 30″ E. at sights.
   171° 17′ 21″ E. at noon.
10. 93° 50′ 45″ W.
11. 44° 25′ 45″ W.
12. 76° 5′ 30″ E.
13. 148° 17′ 45″ W.
14. 179° 17′ 45″ W. at sights.
    179° 46′ 32″ E. at noon.
15. 34° 45′ 0″ W.
16. 36° 51′ 45″ W.
17. 45° 55′ 0″ W.
18. 47° 7′ 0″ W.
19. 74° 21′ 45″ W.
20. 75° 16′ 15″ W.

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